

Wavelets and spectral triples for fractal representations of Cuntz algebras

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March 24, 2016

Abstract

In this article we provide an identification between the wavelet decompositions of certain fractal representations of C^* algebras of directed graphs of M. Marcolli and A. Paolucci [19], and the eigenspaces of Laplacians associated to spectral triples constructed from Cantor fractal sets that are the infinite path spaces of Bratteli diagrams associated to the representations, with a particular emphasis on wavelets for representations of Cuntz C^* -algebras \mathcal{O}_D . In particular, in this setting we use results of J. Pearson and J. Bellissard [20], and A. Julien and J. Savinien [15], to construct first the spectral triple and then the Laplace–Beltrami operator on the associated Cantor set. We then prove that in certain cases, the orthogonal wavelet decomposition and the decomposition via orthogonal eigenspaces match up precisely. We give several explicit examples, including an example related to a Sierpinski fractal, and compute in detail all the eigenvalues and corresponding eigenspaces of the Laplace–Beltrami operators for the equal weight case for representations of \mathcal{O}_D , and in the uneven weight case for certain representations of \mathcal{O}_2 , and show how the eigenspaces and wavelet subspaces at different levels first constructed in [8] are related.

2010 Mathematics Subject Classification: 46L05.

Key words and phrases: Weighted Bratteli diagrams; Ultrametric Cantor set; Spectral triples; Laplace Beltrami operators; Wavelets.

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1 Introduction

In the 2011 paper [19], M. Marcolli and A. Paolucci, motivated by work of A. Jonsson [13] and R. Strichartz [23], studied representations of Cuntz–Krieger C^* -algebras on Hilbert spaces associated to certain fractals, and constructed what they termed “wavelets” in these Hilbert spaces. These wavelets were so called because they provided an orthogonal decomposition of the Hilbert space, and the partial isometries associated to the C^* -algebra in question gave “scaling and translation” operators taking one orthogonal subspace to another. The results of Marcolli and Paolucci were generalized first to certain fractal representations of C^* -algebras associated to directed graphs and then to representations of higher-rank graph C^* -algebras $C^*(\Lambda)$ by some of the authors of this article in [7] and [8]. The k -graph C^* -algebras $C^*(\Lambda)$ of Robertson and Steger [22] are particular examples of these higher-rank graph algebras, and it was shown in [7] that for these Robertson–Steger C^* -algebras there is a faithful representation of $C^*(\Lambda)$ on $L^2(X, \mu)$, where X is a fractal space with Hausdorff measure μ . Moreover, this Hilbert space also admits a wavelet decomposition – that is, an orthogonal decomposition such that the representation of $C^*(\Lambda)$ is generated by “scaling and translation” operators that move between the orthogonal subspaces. As in Marcolli and Paolucci’s original construction, the wavelets in [7] and [8] had a characteristic structure, in that they were chosen to be orthogonal to a specific type of function in the path space that could be easily recognized.

Earlier, the theory of spectral triples and Fredholm modules of A. Connes had generated great interest [5], and such objects had been constructed for dense subalgebras of several different classes of C^* -algebras, including the construction of spectral triples by E. Christensen and C. Ivan on the C^* -algebras of Cantor sets [3], which in turn motivated the work of J. Pearson and J. Bellissard, who constructed spectral triples and related Laplacians on ultrametric Cantor sets [20]. Expanding on the work of Pearson and Bellissard, A. Julien and J. Savinien studied similarly constructed Laplacians on fractal sets constructed from substitution tilings [15]. In both the papers of Pearson and Bellissard and of Julien and Savinien, after the Laplacian operators were described, spanning sets of functions for the eigenspaces of the Laplacian were explicitly described in terms of

differences of characteristic functions.

It became apparent to the authors of the current paper that certain components of the wavelet system as described in [7] and the explicit eigenfunctions given by Julien and Savinien in [15] seemed related, and one of the aims of this paper is to analyze this similarity in the case of C^* -algebras of directed graphs as represented on their infinite path spaces. Indeed, we will show that under appropriate hypotheses, each orthogonal subspace described in the wavelet decomposition of [7] can be expressed as a union of certain of the eigenspaces of the Laplace–Beltrami operator from [15]. We suspect that the hypotheses required for this result can be substantially weakened from their statement in Theorem 5.1 below, and plan to explore this question in future work [9].

More broadly, the goal of this paper is to elucidate the connections between graph C^* -algebras, wavelets on fractals, and spectral triples. We focus here on the case of one particular directed graph, namely the graph Λ_D which has D vertices and, for each pair (v, w) of vertices, a unique edge e with source w and range v . Again, many of the results presented here will hold in greater generality; see the forthcoming paper [9] for details.

In this paper we introduce the graph C^* -algebra (also known as a Cuntz algebra) associated to Λ_D ; discuss the associated representations on fractal spaces as in [19, 7]; and present the associated spectral triples and Laplace–Beltrami operators associated to (fractal) ultrametric Cantor sets as adapted from recent work by Julien–Savinien, Pearson and Bellissard, Christensen et al., see e.g. [15, 20, 4]. In particular we show in Theorem 5.1 that when one constructs the Laplace–Beltrami operator of [15] associated to the infinite path space of Λ_D (which is an ultrametric Cantor set), the wavelets in [19] are exactly the eigenfunctions of the Laplacian. We then compute in detail all the eigenfunctions and eigenvalues of the Laplace–Beltrami operator associated to a representation of the Cuntz algebra \mathcal{O}_D on a Sierpinski type fractal set (see [19] Section 2.6 and Section 3.1 below for the definition of this representation). For several different choices of a measure on the infinite path space of Λ_D , we also compute all the the eigenfunctions and eigenvalues of the associated Laplace–Beltrami operator; in the case when $D = 2$ and this measure arises from assigning the two vertices of Λ_D the weights r and $1 - r$ for some $r \in [0, 1]$, we compare these results to wavelets associated to certain representations of \mathcal{O}_2 analyzed in Section 3 of [8].

In further work [9] we will generalize these constructions to more general directed graphs and to higher-rank graphs, and also explain how to generalize certain other spectral triples associated to directed graphs, such as those described in [2], [3], [10], [24], and [14], to higher-rank graphs.

The structure of the paper is as follows. In Section 2, we review the definition of directed graphs, with an emphasis on finite graphs and the construction of both the infinite path space and Bratteli diagrams associated to finite directed graphs, the first as described in [19] among other places, and the second as described in [21]. When the incidence matrices for our graphs are $\{0, 1\}$ matrices, the infinite path space can be defined in terms of both edges and vertices, and we describe this correspondence, together with the identification of the infinite path space Λ^∞ with the associated infinite path space of the Bratteli diagram $\partial\mathcal{B}_\Lambda$ for a finite directed graph Λ . In so doing, we note that these spaces are Cantor sets. We also review the semibranching function systems of K.

Kawamura [17] and Marcolli and Paolucci [19] in this section, with an emphasis on those systems giving rise to representations of the Cuntz algebras \mathcal{O}_D . In Section 3, we review representations of \mathcal{O}_D on the L^2 -spaces of Sierpinski fractals first constructed by Marcolli and Paolucci in [19], and show that these representations are equivalent to the standard positive monic representations of \mathcal{O}_D defined by D. Dutkay and P. Jorgensen in [6]. In Section 4, we review the construction of spectral triples associated to weighted Bratteli diagrams, described by Pearson and Bellissard in [20] and Julien and Savinien in [15], and provide explicit details of their construction for a variety of weights on the Bratteli diagram $\partial\mathcal{B}_D$ associated to the graph Λ_D . We describe in Theorem 4.9 the conditions under which the measure on $\partial\mathcal{B}_D$ agrees with the measure introduced by Marcolli and Paolucci, which we describe in Section 2. We also introduce the Laplace–Beltrami operator of Pearson and Bellissard [20] in this setting and review the specific formulas for its eigenvalues and associated eigenspaces. In Section 5 we review the construction of Marcolli and Paolucci’s wavelets associated to representations of Cuntz–Krieger C^* -algebras on the L^2 -spaces of certain fractal spaces, with the notation for these subspaces provided in earlier papers [7, 8] with an emphasis on representations of the Cuntz C^* -algebra \mathcal{O}_D , and prove our main theorem (Theorem 5.1), which is that in all cases that we consider, the wavelet subspaces for Marcolli and Paolucci’s representations can be identified with the eigenspaces of the Laplace–Beltrami operator associated to the related Bratteli diagram. In Section 6, we examine certain representations of \mathcal{O}_D where the weights involved are unevenly distributed among the vertices of Λ_D , and specializing to the study of uneven weights associated to representations of \mathcal{O}_2 , we compute explicitly the associated eigenvalues and eigenspaces for the Laplace–Beltrami operator in this case, and provide the correspondence between these eigenspaces and certain wavelet spaces for monic representations of \mathcal{O}_2 first computed in [8].

This work was partially supported by a grant from the Simons Foundation (#316981 to Judith Packer).

2 Cantor sets associated to directed graphs

We begin with a word about conventions. Throughout this paper, \mathbb{N} consists of the positive integers, $\mathbb{N} = \{1, 2, 3, \dots\}$; we use \mathbb{N}_0 to denote $\{0, 1, 2, 3, \dots\}$. The symbol \mathbb{Z}_N indicates the set $\{0, \dots, N-1\}$.

The Bratteli diagrams we discuss below do not have a root vertex; indeed, we think of the edges in a Bratteli diagram as pointing towards the zeroth level of the diagram. See Remark 2.5 for more details.

2.1 Directed graphs and Bratteli diagrams

Definition 2.1. A *directed graph* Λ consists of a set of vertices Λ^0 and a set of edges Λ^1 and range and source maps $r, s : \Lambda^1 \rightarrow \Lambda^0$. We say that Λ is *finite* if

$$\Lambda^n = \{e_1 e_2 \dots e_n : e_i \in \Lambda^1, r(e_i) = s(e_{i-1}) \forall i\}$$

is finite for all $n \in \mathbb{N}$. If $\gamma = e_1 \cdots e_n$, we define $r(\gamma) = r(e_1)$ and $s(\gamma) = s(e_n)$, and we write $|\gamma| = n$. By convention, a path of length 0 consists of a single vertex (no edge): if $|\gamma| = 0$ then $\gamma = (v)$ for some vertex v .

We say that Λ has *no sources* if $v\Lambda^n = \{\gamma \in \Lambda^n : r(\gamma) = v\} \neq \emptyset$ for all $v \in \Lambda^0$ and all $n \in \mathbb{N}$. We say that Λ is *strongly connected* if

$$v\Lambda w = \bigcup_{n \in \mathbb{N}} \{\gamma \in \Lambda^n : s(\gamma) = w\} \neq \emptyset$$

for all $v, w \in \Lambda^0$. In a slight abuse of notation, if Λ^n denotes the set of finite paths of length n , we denote by $\Lambda = \bigcup_{n \in \mathbb{N}_0} \Lambda^n$ the set of all finite paths, and by Λ^∞ the set of infinite paths of a finite directed graph Λ :

$$\Lambda^\infty = \left\{ (e_i)_{i \in \mathbb{N}} \in \prod_{i=1}^{\infty} \Lambda^1 : s(e_i) = r(e_{i+1}) \ \forall i \in \mathbb{N} \right\}.$$

For $\gamma \in \Lambda$, we write $[\gamma] \subseteq \Lambda^\infty$ for the set of infinite paths with initial segment γ :

$$[e_1 \dots e_n] = \{(f_i)_i \in \Lambda^\infty : f_i = e_i \ \forall 1 \leq i \leq n\}. \quad (1)$$

We say that a path $\gamma = e_1 \dots e_n$ has length n and write $|\gamma| = n$. If $\gamma = (v)$ is a path of length 0, then $[\gamma] = [v] = \{(f_i)_i \in \Lambda^\infty : r(f_1) = v\}$.

Given a finite directed graph Λ , the *vertex matrix* A of Λ is an $\Lambda^0 \times \Lambda^0$ matrix with entry $A(v, w) = |v\Lambda^1 w|$ counting the number of edges with range v and source w in Λ .

Remark 2.2. As shown in [18] Corollary 2.2, if Λ is finite and source-free, the cylinder sets $\{[\gamma] : \gamma \in \Lambda\}$ form a compact open basis for a locally compact, totally disconnected, Hausdorff topology on Λ^∞ .¹ If Λ is finite, Λ^∞ is also compact.

According to [11] Proposition 8.1, a strongly connected finite directed graph Λ has a distinguished Borel measure M on the infinite path space Λ^∞ which is given in terms of the spectral radius $\rho(A)$ of the vertex matrix A ;

$$M([\gamma]) = \rho(A)^{-|\gamma|} P_{s(\gamma)}, \quad (2)$$

where $(P_v)_{v \in \Lambda^0}$ is the unimodular Perron–Frobenius eigenvector of the vertex matrix A . (See section 2 of [7] for details).

Definition 2.3. Let Λ be a finite directed graph with no sources. The *Bratteli diagram* associated to Λ is an infinite directed graph \mathcal{B}_Λ , with the set of vertices $V = \sqcup_{n \geq 0} V_n$ and the set of edges $E = \sqcup_{n \geq 1} E_n$ such that

- (a) For each $n \in \mathbb{N}_0$, $V_n \cong \Lambda^0$ and $E_{n+1} \cong \Lambda^1$.
- (b) There are a range map and a source map $r, s : E \rightarrow V$ such that $r(E_n) \subseteq V_{n-1}$ and $s(E_n) \subseteq V_n$ for all $n \in \mathbb{N}$.

¹Note that if Λ is finite, it is also row-finite, according to the definition given in Section 2 of [18].

A *path* γ of length $n \in \mathbb{N}$ in \mathcal{B}_Λ is an element

$$e_1 e_2 \dots e_n = (e_1, e_2, \dots, e_n) \in \prod_{i=1}^n E_n$$

which satisfies $|e_i| = 1 \ \forall i$, and $s(e_i) = r(e_{i+1})$ for all $1 \leq i \leq n-1$. We denote by $F\mathcal{B}_\Lambda$ the set of all finite paths in the Bratteli diagram \mathcal{B}_Λ , and by $F^n\mathcal{B}_\Lambda$ the set of all finite paths in the Bratteli diagram \mathcal{B}_Λ of length n .

We denote by $\partial\mathcal{B}_\Lambda$ the set of infinite paths in the Bratteli diagram \mathcal{B}_Λ ;

$$\partial\mathcal{B}_\Lambda = \{e_1 e_2 \dots = (e_1, e_2, \dots) \in \prod_{n=1}^{\infty} E_n : |e_i| = 1, s(e_i) = r(e_{i+1}) \ \forall i \in \mathbb{N}\}.$$

Given a (finite or infinite) path $\gamma = e_1 e_2 \dots$ in \mathcal{B}_Λ and $m \in \mathbb{N}$, we write

$$\gamma[0, m] = e_1 e_2 \dots e_m.$$

If $m = 0$ we write $\gamma[0, 0] = r(\gamma)$.

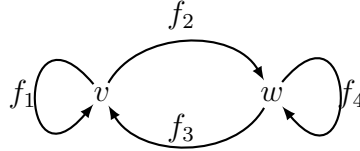
Remark 2.4. Any finite path γ of a length n in a directed graph (or a Bratteli diagram) is given by a string of n edges $e_1 e_2 \dots e_n$, which can be written uniquely as a string of vertices $v_0 v_1 \dots v_n$ such that $r(e_i) = v_{i-1}$ and $s(e_i) = v_i$ for $1 \leq i \leq n$. Conversely, if the vertex matrix A has all entries either 0 or 1 (as will be the case in all of our examples), a given string of vertices $v_0 v_1 \dots v_n$ with $v_i \in V_n$ for all $n \in \mathbb{N}_0$ corresponds to at most one string of edges, and hence at most one finite path γ . Thus even though our formal definition of a path is given as a string of edges, sometimes we use the notation of a string of vertices for a path.

Remark 2.5. Note that our description of a Bratteli diagram is different from the one in [15] and [1]. First, the edges in E_n in [15] and in [1] have source in V_n and range in V_{n+1} ; in other words, they point in the opposite direction from our edges. More substantially, though, in [15] and [1] every finite (or infinite) path in a Bratteli diagram starts from a vertex called a root vertex, \circ , and any finite path that ends in V_n is given by $\epsilon_{r(e_1)} e_1 e_2 \dots e_n$, where for each vertex $v \in V_0$, there is a unique edge ϵ_v connecting \circ and v . This implies that a finite path that ends in V_n consists of $n+1$ edges in their Bratteli diagram. However, our description of a Bratteli diagram in Definition 2.3 does not include a root vertex, and a finite path that ends in V_n consists of n edges. Thus, when we discuss Theorem 4.3 of [15] in Sections 4.2 and 6 below, we will need to introduce a single path, the “empty path” of length -1, which we will denote by $\gamma[0, -1]$ for any and all paths $\gamma \in F\mathcal{B}_\Lambda$. The cylinder set of this path is $[\circ] = \partial\mathcal{B}_\Lambda$ when we translate Theorem 4.3 of [15] to our setting.

Remark 2.6. As is suggested by the notation, a finite directed graph and its associated Bratteli diagram encode the same information in their sets of finite and infinite paths. We wish to emphasize this correspondence in this paper, to illuminate the way tools from a variety of disciplines combine to give us information about wavelets on fractals.

Remark 2.7. If Λ is a strongly connected finite directed graph, then Λ has no sources by Lemma 2.1 of [11]. Hence every vertex of the associated Bratteli diagram \mathcal{B}_Λ also receives an edge.

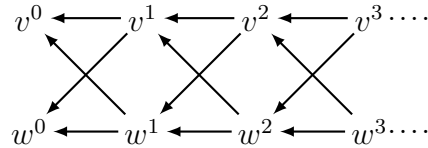
Example 2.8. Consider a directed graph Λ with two vertices v, w and four edges f_1, f_2, f_3 and f_4 given as follows:



Note that Λ is finite and strongly connected, and (consequently) has no sources. The vertex matrix A is given by

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

and the associated Bratteli diagram \mathcal{B}_Λ is



Proposition 2.9. *Let Λ be a finite directed graph. If every vertex v in the directed graph Λ receives two distinct infinite paths, then Λ^∞ (equivalently, $\partial\mathcal{B}_\Lambda$) has no isolated points and hence it is a Cantor set.*

Proof. Recall that a Cantor set is a totally disconnected, compact, perfect topological space. Moreover, Λ^∞ is always compact Hausdorff and totally disconnected by Corollary 2.2 of [18], so it will suffice to show that Λ^∞ has no isolated points.

Suppose Λ^∞ has an isolated point $(e_i)_{i \in \mathbb{N}}$. Since the cylinder sets form a basis for the topology on Λ^∞ , this implies that there exists $n \in \mathbb{N}$ such that $[e_1 \cdots e_n]$ only contains $(e_i)_{i \in \mathbb{N}}$. In other words, for each $m \geq n$, there is only one infinite path with range $s(e_m)$, contradicting the hypothesis of the proposition. \square

Corollary 2.10. *If Λ is a finite directed graph with $\{0, 1\}$ vertex matrix A and every row sum of A is at least 2, then Λ^∞ (equivalently, $\partial\mathcal{B}_\Lambda$) is a Cantor set.*

Proof. Note that the sum of the v th row of A represents the number of edges in Λ with range v . If every vertex receives at least two edges, then any cylinder set $[\gamma]$ will contain infinitely many elements, so Λ^∞ has no isolated points. \square

Corollary 2.10 tells us that the infinite path space of Example 2.8 is a Cantor set.

2.2 Cuntz algebras and representations on fractal spaces

Definition 2.11 ([6, Definition 2.1]). Fix an integer $D > 1$. The *Cuntz algebra* \mathcal{O}_D is the universal C^* -algebra generated by isometries $\{T_i\}_{i=0}^{D-1}$ satisfying the Cuntz relations

$$T_j^* T_i = \delta_{ij} I, \quad (3)$$

and

$$\sum_{i=0}^{D-1} T_i T_i^* = I. \quad (4)$$

The above definition of \mathcal{O}_D is equivalent to the definition of \mathcal{O}_{A_D} in the beginning of section 2 of [19] associated to the matrix A_D that is a $D \times D$ matrix with 1 in every entry:

$$A_D = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & \dots & 1 \end{pmatrix}. \quad (5)$$

As had been done previously by K. Kawamura [17], Marcolli and Paolucci constructed representations of \mathcal{O}_D (and more generally, the Cuntz–Krieger algebras \mathcal{O}_A associated to a matrix A) by employing the method of “semibranching function systems.” We note for completeness that the semibranching function systems of Kawamura [17] were for the most part defined on finite Euclidean spaces, e.g. the unit interval $[0, 1]$, whereas the semibranching function systems used by Marcolli and Paolucci [19] were mainly defined on Cantor sets.

Definition 2.12 (cf. [17], [19, Definition 2.1], [1, Definition 2.16]). Let (X, μ) be a measure space, fix an integer $D > 1$ and let $\{\sigma_i : X \rightarrow X\}_{i \in \mathbb{Z}_D}$ be a collection of μ -measurable maps. The family of maps $\{\sigma_i\}_{i \in \mathbb{Z}_D}$ is called a *semibranching function system* on (X, μ) with coding map $\sigma : X \rightarrow X$ if the following conditions hold:

1. For $i \in \mathbb{Z}_D$, set $R_{[i]} = \sigma_i(X)$. Then we have

$$\mu(X \setminus \cup_{i \in \mathbb{Z}_D} R_{[i]}) = 0 \quad \text{and} \quad \mu(R_{[i]} \cap R_{[j]}) = 0 \quad \text{for } i \neq j.$$

2. For $i \in \mathbb{Z}_D$, we have $\mu \circ \sigma_i \ll \mu$ and the Radon–Nikodym derivative satisfies

$$\frac{d(\mu \circ \sigma_i)}{d\mu} > 0, \quad \mu\text{-a.e.} \quad (6)$$

3. For all $i \in \mathbb{Z}_D$, we have

$$\sigma \circ \sigma_i(x) = x, \quad \mu\text{-a.e.}$$

Kawamura and then Marcolli and Paolucci observed the following relationship between semibranching function systems and representations of \mathcal{O}_D :

Proposition 2.13 (cf. [19, Proposition 2.4], [6, Theorem 2.22]). *Let (X, μ) be a measure space, and let $\{\sigma_i : X \rightarrow X\}_{i \in \mathbb{Z}_D}$ be a semibranching function system on (X, μ) with coding map $\sigma : X \rightarrow X$. For each $i \in \mathbb{Z}_D$ define $S_i : L^2(X, \mu) \rightarrow L^2(X, \mu)$ by*

$$S_i(\xi)(x) = \chi_{R_{[i]}}(x) \left(\frac{d\mu \circ \sigma_i}{d\mu}(\sigma(x)) \right)^{-\frac{1}{2}} \xi(\sigma(x)) \text{ for } \xi \in L^2(X, \mu) \text{ and } x \in X.$$

Then the family $\{S_i\}_{i \in \mathbb{Z}_D}$ satisfies the Cuntz relations Equations (3) and (4), and therefore generates a representation of the Cuntz algebra \mathcal{O}_D .

Example 2.14. Let Λ_D be the directed graph associated to the vertex matrix A_D . We can define a semibranching function system $\{(\sigma_i)_{i \in \mathbb{Z}_D}, \sigma\}$ on the Cantor set (Λ_D^∞, M) by thinking of elements of Λ_D^∞ as sequences of vertices $(v_i)_{i \in \mathbb{N}_0}$ with $v_j \in \mathbb{Z}_D \forall j$. With this convention, we set

$$\sigma_i(v_0 v_1 v_2 \dots) = (i v_0 v_1 v_2 \dots) \text{ and } \sigma(v_0 v_1 \dots) = (v_1 v_2 \dots).$$

Then the Radon–Nikodym derivative $\frac{d(M \circ \sigma_i)}{dM}$ is given by

$$\frac{d(M \circ \sigma_i)}{dM} = \frac{1}{D}$$

since the cylinder set $R_{[i]}$ has measure $\frac{1}{D}$ for all i , and the associated operators S_i are given by

$$S_i(\xi)(v_0 v_1 v_2 \dots) = \begin{cases} \sqrt{D} \xi(v_1 v_2 \dots) & \text{if } v_0 = i \\ 0 & \text{else.} \end{cases}$$

This representation of \mathcal{O}_D is faithful by Theorem 3.6 of [7], since every cycle in Λ_D has an entrance.

Example 2.15 (cf. [19, Proposition 2.6]). Take an integer $D > 1$, and let $K_D = \prod_{j=1}^\infty [\mathbb{Z}_D]_j$, which is called the *Cantor group* on D letters in Definition 2.3 of [6]. As described in Section 2 of [8], K_D has a Cantor set topology which is generated by cylinder sets

$$[n] = \{(i_j)_{j=1}^\infty \in K_D : i_1 = n\}.$$

According to Section 3 of [6], there is a measure ν_D on K_D given by

$$\nu_D([n_1 n_2 \dots n_m]) = \prod_{j=1}^m \frac{1}{D} = \frac{1}{D^m}.$$

Note that ν_D is a Borel measure on K_D with respect to the cylinder-set Cantor topology.

For each $j \in \mathbb{Z}_D$, define σ_j on K_D by

$$\sigma_j((i_1 i_2 \dots i_k \dots)) = (j i_1 i_2 \dots i_k \dots).$$

Then

$$R_{[j]} = \sigma_j(K_D) = \{(j i_1 i_2 \dots i_k \dots) : (i_1 i_2 \dots i_k \dots) \in K_D\} = [j],$$

and, denoting by σ the one-sided shift on K_D , $\sigma((i_1 i_2 \cdots i_k \cdots)) = (i_2 i_3 \cdots i_{k+1} \cdots)$, we have that $\sigma \circ \sigma_j(x) = x$ for all $x \in K_D$ and $j \in \mathbb{Z}_D$. Marcolli and Paolucci show in Section 2.1 of [19] that this data gives a semibranching function system. Moreover, since the measure of each set $R_{[i]}$ is $\frac{1}{D}$, the Radon–Nikodym derivative $\frac{d(\nu_D \circ \sigma_i)}{d\nu_D}$ satisfies

$$\frac{d(\nu_D \circ \sigma_i)}{d\nu_D} = \frac{1}{D}.$$

Thus, Proposition 2.13 implies that there is a family of operators $\{S_i\}_{i \in \mathbb{Z}_D} \subseteq B(L^2(K_D, \nu_D))$ that generates a representation of the Cuntz algebra \mathcal{O}_D .

Moreover, this representation is faithful by Theorem 3.6 of [7]. To see this, let Λ_D denote the directed graph with vertex matrix A_D , and note that labeling the vertices of Λ_D by $\{0, 1, \dots, D-1\}$ allows us to identify an infinite path $(e_i)_{i \in \mathbb{N}} \in \partial \mathcal{B}_D$ with the sequence $(r(e_i))_{i \in \mathbb{N}} \in K_D$. Moreover, in this case the Perron–Frobenius eigenvector associated to A_D is

$$P = \left(\frac{1}{D}, \frac{1}{D}, \dots, \frac{1}{D} \right),$$

and consequently

$$M([e_1 \dots e_n]) = \frac{1}{D^{n+1}} = \nu_D([r(e_1)r(e_2) \cdots r(e_n)s(e_n)]).$$

Since the cylinder sets generate the topology on both K_D and on $\partial \mathcal{B}_D$, this identification is measure-preserving. Thus, the representation $\{S_i\}_{i \in \mathbb{Z}_D}$ of \mathcal{O}_D on $L^2(K_D, \nu_D)$ is equivalent to the infinite path representation of Example 2.14. We can apply Theorem 3.6 of [7] to this latter representation to conclude that it is faithful, since every cycle in the graph Λ_D associated to A_D has an entry. In other words,

$$C^*(\{S_i\}_{i \in \mathbb{Z}_D}) \cong \mathcal{O}_D.$$

3 The action of \mathcal{O}_D on $L^2(\mathbb{S}_A, H)$

As mentioned in the Introduction, we wish to show that when we represent \mathcal{O}_D on a 2-dimensional Sierpinski fractal \mathbb{S}_A , this representation of \mathcal{O}_D also gives rise to wavelets. We will then compare these wavelets with the eigenfunctions of the Laplace–Beltrami operator Δ_s of [15] that is associated to A_D , the $D \times D$ matrix of all 1’s (that is, the matrix associated to the Cuntz algebra \mathcal{O}_D). To compare these functions, we will establish a measure-preserving isomorphism between \mathbb{S}_A and the infinite path space of the directed graph (equivalently, Bratteli diagram) associated to \mathcal{O}_D in this section. (See Theorem 3.1 below).

3.1 The Sierpinski fractal representation for \mathcal{O}_D

Let N and D be positive integers with $N \geq 2$, and let A be a $N \times N\{0, 1\}$ -matrix with exactly D entries consisting of the number 1. Suppose that the nonzero entries of A are in

positions $\{(a_j, b_j)\}_{j=0}^{D-1}$, where $a_j, b_j \in \{0, 1, \dots, N-1\}$ and in a lexicographic ordering we have $(a_0, b_0) < (a_1, b_1) < \dots < (a_{D-1}, b_{D-1})$. Here we say $(a, b) < (a', b')$ if either $a < a'$ or if $a = a'$ and $b < b'$.

In Section 2.6 of [19], Marcolli and Paolucci defined the Sierpinski fractal associated to A , $\mathbb{S}_A \subset [0, 1]^2$, as follows:

$$\mathbb{S}_A = \left\{ (x, y) = \left(\sum_{i=1}^{\infty} \frac{x_i}{N^i}, \sum_{i=1}^{\infty} \frac{y_i}{N^i} \right) : x_i, y_i \in \mathbb{Z}_N, A_{x_i, y_i} = 1, \forall i \in \mathbb{N} \right\}.$$

For each $j \in \mathbb{Z}_D$, we define

$$\tau_j(x, y) = \left(\frac{x}{N} + \frac{a_j}{N}, \frac{y}{N} + \frac{b_j}{N} \right) \quad \text{and} \quad \tau(x, y) = \left(N \left(x - \frac{x_1}{N} \right), N \left(y - \frac{y_1}{N} \right) \right). \quad (7)$$

Lemma 2.23 of [19] tells us that the operators $\{\tau_j\}_{j \in \mathbb{Z}_D}$ form a semibranching function system with coding map τ , and hence determine a representation of the Cuntz algebra \mathcal{O}_D associated to A_D given in (5), on the Hilbert space $L^2(\mathbb{S}_A, H)$. Here H is the Hausdorff measure on the fractal \mathbb{S}_A .

According to the work of Hutchinson [12], we have

$$\mathbb{S}_A = \bigcup_{i=1}^D \tau_j(\mathbb{S}_A).$$

Moreover, the work of [12] shows that the Hausdorff measure H on \mathbb{S}_A is the unique Borel probability measure on \mathbb{S}_A satisfying the self-similarity equation

$$H = \sum_{i=0}^{D-1} \frac{1}{D} (\tau_j)_*(H). \quad (8)$$

In other words,

$$H(\tau_j(\mathbb{S}_A)) = \frac{1}{D} H(\mathbb{S}_A) = \frac{1}{D}.$$

It follows that, since

$$\begin{aligned} \tau_j(\mathbb{S}_A) &= \left\{ \left(\sum_{i=1}^{\infty} \frac{x_i}{N^i}, \sum_{i=1}^{\infty} \frac{y_i}{N^i} \right) : (x_1, y_1) = (a_j, b_j) \right\}, \\ H \left(\left\{ \sum_{i=1}^{\infty} \frac{x_i}{N^i}, \sum_{i=1}^{\infty} \frac{y_i}{N^i} \right\} \in \mathbb{S}_A : (x_1, y_1) = (a_j, b_j) \right) &= \frac{1}{D}. \end{aligned}$$

By repeatedly applying the measure-similitude equation (8) we obtain

$$\begin{aligned} H \left(\left\{ \left(\sum_{i=1}^{\infty} \frac{x_i}{N^i}, \sum_{i=1}^{\infty} \frac{y_i}{N^i} \right) \in \mathbb{S}_A : \forall 1 \leq i \leq M, (x_i, y_i) = (a_{j_i}, b_{j_i}) \right\} \right) \\ = H(\tau_{j_1} \circ \tau_{j_2} \circ \dots \circ \tau_{j_M}(\mathbb{S}_A)) = \left(\frac{1}{D} \right)^M. \quad (9) \end{aligned}$$

3.2 The measure-preserving isomorphism

In this section, we discuss in more detail the relationship between the representation of \mathcal{O}_D on $L^2(\mathbb{S}_A, H)$ and the infinite path representation of \mathcal{O}_D on $L^2(\partial\mathcal{B}_D, M)$ described in Example 2.14.

First, we note that the Hausdorff dimension of the Sierpinski fractal \mathbb{S}_A introduced above is

$$\frac{\ln D}{\ln N},$$

as established in Hutchinson's paper [12].² In particular, in the classical case of the Sierpinski triangle corresponding to the 2×2 matrix $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, the Hausdorff dimension of \mathbb{S}_A is $\frac{\ln 3}{\ln 2}$.

The main goal of this section is to prove the following:

Theorem 3.1. *Let A be the $N \times N$ matrix with entries consisting of only 0's and 1's with D incidences of 1's in the entries*

$$(a_0, b_0) < (a_1, b_1) < \cdots < (a_{D-1}, b_{D-1}),$$

where $a_j, b_j \in \mathbb{Z}_N$. Consider the Sierpinski gasket fractal

$$\mathbb{S}_A = \left\{ \left(\sum_{i \in \mathbb{N}} \frac{x_i}{N^i}, \sum_{i \in \mathbb{N}} \frac{y_i}{N^i} \right) : A(x_i, y_i) = 1, \forall i \in \mathbb{N} \right\}.$$

Then there is a measure-theoretic isomorphism

$$\Upsilon = \Phi \circ \Theta : (\partial\mathcal{B}_D, M) \rightarrow (\mathbb{S}_A, H),$$

where $(\partial\mathcal{B}_D, M)$ is the infinite path space of the Bratteli diagram associated to the $D \times D$ matrix with all ones, and M is the measure given by Equation (2):

$$M[\gamma] = D^{-|\gamma|-1}.$$

Moreover, if $\{S_i\}_{i \in \mathbb{Z}_D}$ denotes the infinite path representation of \mathcal{O}_D on $(\partial\mathcal{B}_A, M)$, and $\{T_i\}_{i \in \mathbb{Z}_D}$ denotes the representation of \mathcal{O}_D on (\mathbb{S}_A, H) associated to the semibranching function system (7), then for all $i \in \mathbb{Z}_D$,

$$T_i = S_i \circ \Upsilon.$$

Proof. Let S_A denote the D -element symbol space of pairs from \mathbb{Z}_N with 1's in the corresponding entry of A :

$$S_A = \{(a_0, b_0), (a_1, b_1), (a_2, b_2), \dots, (a_{D-1}, b_{D-1})\} \subset \mathbb{Z}_N \times \mathbb{Z}_N,$$

²This formula is not in line with [19, Equation (2.64)], which gives $\ln D / (2 \ln N)$ for the Hausdorff dimension. However, said equation appears to be a typo: the dimension should be 2 when $D = N^2$ (i.e. when \mathcal{S}_A is the unit square).

and let X_A be the infinite product space $X_A = \prod_{i=1}^{\infty} S_A$. Giving S_A the discrete topology and X_A the product topology, we see that X_A is a Cantor set, by the arguments of Section 2 of [8]. For every $i \in \mathbb{N}$, let $\mu_{i,A}$ be the normalized counting measure on S_A ; that is, for $S \subset S_A$,

$$\mu_{i,A}(S) = \frac{\#(S)}{D},$$

and let μ_A denote the infinite product measure $\mu_A = \prod_{i=1}^{\infty} (\mu_{i,A})$. Note if we let

$$[(a_{j_1}, b_{j_1})(a_{j_2}, b_{j_2}) \cdots (a_{j_M}, b_{j_M})] \quad (10)$$

denote the cylinder set

$$\begin{aligned} & [(a_{j_1}, b_{j_1})(a_{j_2}, b_{j_2}) \cdots (a_{j_M}, b_{j_M})] \\ &= \{((x_i, y_i))_{i=1}^{\infty} \in X_A : (x_i, y_i) = (a_{j_i}, b_{j_i}) \ \forall \ 1 \leq i \leq M\}, \end{aligned}$$

then

$$\mu_A([(a_{j_1}, b_{j_1})(a_{j_2}, b_{j_2}) \cdots (a_{j_M}, b_{j_M})]) = \frac{1}{D^M}.$$

Define now a map $\Phi : X_A \rightarrow \mathbb{S}_A$ by

$$\Phi(((x_i, y_i))_{i=1}^{\infty}) = \left(\sum_{i=1}^{\infty} \frac{x_i}{N^i}, \sum_{i=1}^{\infty} \frac{y_i}{N^i} \right).$$

The map Φ is continuous from the product topology on X_A to the topology on \mathbb{S}_A inherited from the Euclidean topology on $[0, 1] \times [0, 1]$. The map Φ is **not** one-to-one, but if we let $E \subset X_A$ denote the set of points on which Φ is not injective, $\mu_A(E) = 0$. Indeed, let's examine the set of points of X_A where Φ may not be one-to-one: non-injectivity can come from pairs of sequence of the forms $(x_i, y_i)_i, (x'_i, y'_i)_i$ where x_i is eventually $N - 1$ and x'_i is eventually 0, and similarly exchanging x and y . Notice also that if A has no ones either on the first or on the last row, there will be no such pairs for which x_i is eventually $N - 1$ and x'_i is eventually 0. Therefore, since A has D total entries equaling 1, if two such pairs $(x_i, y_i)_i$ and $(x'_i, y'_i)_i$ are going to have the same image under Φ , there need to be at most $D - 1$ ones on the first row, and the same on the last row. Therefore, the measure of the set of pairs $(x_i, y_i)_i$ for which x_i is eventually $N - 1$ is smaller than $[(D - 1)/D]^n$ for all n : it has zero measure. We reason similarly for the set of pairs $(x_i, y_i)_i$ for which x_i is eventually 0, for which y_i is eventually 0 and for which y_i is eventually $N - 1$. In conclusion, the set of points in X_A on which Φ has a risk of not being one-to-one has measure zero.

We also note that since Φ is continuous, it is a Borel measurable map, and that for any Borel subset B of \mathbb{S}_A ,

$$\mu_A \circ [\Phi]_*(B) = H(B).$$

This is the case because a length- M cylinder set in \mathbb{S}_A (that is, any cylinder set $[(x_1, y_1), \dots, (x_M, y_M)]$ consisting of all points in \mathbb{S}_A whose first M pairs of N -adic digits

are fixed) has H -measure $\frac{1}{D^M}$, whereas when one pulls such sets back via Φ , we obtain cylinder sets of the form

$$[(a_{j_1}, b_{j_1})(a_{j_2}, b_{j_2}) \cdots (a_{j_M}, b_{j_M})] \subseteq X_A$$

which also have measure D^{-M} . Since these sets generate the Borel σ -algebras for \mathbb{S}_A and X_A respectively, we get the desired equality of the measures.

Now let \mathcal{B}_D be the Bratteli diagram with D vertices at each level, associated to the matrix A_D given in (5) (and, hence, to the directed graph Λ_D with D vertices and all possible edges). We equip the infinite path space $\partial\mathcal{B}_D$ with the measure of Equation (2), which in this case is $M([\gamma]) = D^{-|\gamma|-1}$. Label the vertices of Λ^0 by $\mathbb{Z}_D = \{0, 1, \dots, D-1\}$, and define $\Theta : \partial\mathcal{B}_D \rightarrow X_A$ by

$$\Theta((e_i)_{i \geq 1}) = ((a_{r(e_1)}, b_{r(e_1)}), (a_{r(e_2)}, b_{r(e_2)}), (a_{r(e_3)}, b_{r(e_3)}), \dots);$$

in other words, Θ takes an infinite path (written in terms of edges) $(e_i)_{i \in \mathbb{N}}$ to the sequence of vertices $(r(e_i))_{i \in \mathbb{N}}$ it passes through, and then maps this sequence of vertices to the corresponding element of X_A . The map Θ is bijective, since each pair of vertices has exactly one edge between them. In addition, both Θ and Θ^{-1} are continuous, since both the topology on $\partial\mathcal{B}_D$ and the topology on X_A are generated by cylinder sets. In other words, Θ is a homeomorphism, and $M = \mu_A \circ [\Theta]_*$.

We thus have shown that $\Upsilon = \Phi \circ \Theta$ is a Borel measure-theoretic isomorphism between the measure spaces $(\partial\mathcal{B}_D, M)$ and (\mathbb{S}_A, H) . A routine computation, using the fact that

$$\Upsilon((e_i)_{i \in \mathbb{N}}) = \left(\sum_{i \in \mathbb{N}} \frac{a_{r(e_i)}}{N^i}, \sum_{i \in \mathbb{N}} \frac{b_{r(e_i)}}{N^i} \right),$$

will show that for any $i \in \mathbb{Z}_D$, $T_i = S_i \circ \Upsilon$ to finish the proof. \square

We now recall the definition of Dutkay and Jorgensen [6] of a *monic* representation of \mathcal{O}_D :

Definition 3.2 (cf. [6, Definition 2.6]). Let $D \in \mathbb{N}$, and let K_D be the infinite product Cantor group defined earlier. Let $\sigma_i : K_D \rightarrow K_D$, $0 \leq i \leq D-1$ be as in Example 2.15. A *nonnegative monic system* is a pair $(\mu, (f_i)_{i \in \mathbb{Z}_D})$ where μ is a Borel probability measure on K_D and $(f_i)_{i \in \mathbb{Z}_D}$ are nonnegative Borel measurable functions in $L^2(K_D, \mu)$ such that $\mu \circ \sigma_i^{-1} \ll \mu$, and such that for all $i \in \mathbb{Z}_D$

$$\frac{d(\mu \circ \sigma_i^{-1})}{d\mu} = (f_i)^2$$

with the property that $f_i(x) \neq 0$, μ a.e. on $\sigma_i(K_D)$, $\forall i \in \mathbb{Z}_D$.

By Equation (2.9) of [6], there is a natural representation of \mathcal{O}_D on $L^2(K_D, \mu)$ associated to a monic system $(\mu, (f_i)_{i \in \mathbb{Z}_D})$ given by

$$\tilde{S}_i f = f_i(f \circ \sigma), \quad (i \in \mathbb{Z}_D, f \in L^2(K_D, \mu)).$$

If $\mu = \nu_D$ and we set $f_i = \sqrt{D} \chi_{\sigma_i(K_D)}$, the corresponding monic system is called the *standard positive monic system* for \mathcal{O}_D .

Corollary 3.3. *The representation of \mathcal{O}_D on $L^2(\mathbb{S}_A, H)$ described in Section 3.1 above is equivalent to the monic representation of \mathcal{O}_D corresponding to the standard positive monic system on $L^2(K_D, \nu_D)$.*

Proof. Theorem 3.1, combined with the measure-theoretic identification of (K_D, ν_D) and $(\partial\mathcal{B}_D, M)$ established in Example 2.15, implies that we have a measure-theoretic isomorphism between (K_D, ν_D) and (\mathbb{S}_A, H) . Thus, to show that the corresponding representations of \mathcal{O}_D are unitarily equivalent, it only remains to check that the operators $\tilde{S}_i = f_i(f \circ \sigma)$ associated to the standard positive monic system, and the operators $\{T_i\}_{i \in \mathbb{Z}_D}$, match up correctly. To that end, observe that

$$\begin{aligned} \tilde{S}_i(\xi)(v_0 v_1 \dots) &= f_i(v_0 v_1 \dots) \xi(v_1 v_2 \dots) = \begin{cases} \sqrt{D} \xi(v_1 v_2 \dots) & \text{if } v_0 = i \\ 0 & \text{else.} \end{cases} \\ &= S_i(\xi)(v_0 v_1 \dots). \end{aligned}$$

Since Theorem 3.1 established that the operators S_i and T_i are unitarily equivalent, the Corollary follows. \square

4 Spectral triples and Laplacians for Cuntz algebras

Let A_D be the $D \times D$ matrix with 1 in every entry and consider the Bratteli diagram \mathcal{B}_D associated to A_D . If $D \geq 2$, then every row sum of A_D is at least 2 by construction, and hence the associated infinite path space of the Bratteli diagram, $\partial\mathcal{B}_D$, is a Cantor set. In this section, by using the methods in [15], we will construct a spectral triple on $\partial\mathcal{B}_D$. This spectral triple gives rise to a Laplace–Beltrami operator Δ_s on $L^2(\partial\mathcal{B}_D, \mu_D)$, where μ_D is the measure induced from the Dixmier trace of the spectral triple as in Theorem 4.9 below. We also compute explicitly the orthogonal decomposition of $L^2(\partial\mathcal{B}_D, \mu_D)$ in terms of the eigenfunctions of the Laplace–Beltrami operator Δ_s (cf. [15, Theorem 4.3]).

4.1 The Cuntz algebra \mathcal{O}_D and its Sierpinski spectral triple

Definition 4.1. Let Λ be a finite directed graph; let $F(\mathcal{B}_\Lambda)_\circ$ be the set of all finite paths on the associated Bratteli diagram, including the empty path whose length we set to -1 by convention.. A *weight* on \mathcal{B}_Λ (equivalently, on Λ) is a function $w : F(\mathcal{B}_\Lambda)_\circ \rightarrow (0, \infty)$ satisfying

(a) $w(\circ) = 1$

(b)

$$\lim_{n \rightarrow \infty} \sup \{w(\eta) : \eta \in \Lambda^n = F^n \mathcal{B}_\Lambda\} = 0,$$

where we denoted by $\Lambda^n = F^n \mathcal{B}_\Lambda$ the set of finite paths of length n on Λ (equivalently, \mathcal{B}_Λ).

(c) For any finite paths η, ν with $s(\eta) = r(\nu)$, we have $w(\eta\nu) < w(\eta)$.

A Bratteli diagram \mathcal{B}_Λ with a weight w is called a *weighted Bratteli diagram*.

Remark 4.2. Observe that a weight that satisfies Definition 2.9 of [15] on the vertices of a Bratteli diagram \mathcal{B}_Λ induces a weight on the finite paths of the Bratteli diagram as in Definition 4.1 above. In fact in [15] and [20] the authors define a weight on $F\mathcal{B}_\Lambda$ by defining the weight first on vertices, and then extending it to finite paths via the formula $w(\eta) = w(s(\eta))$, for $\eta \in F\mathcal{B}_\Lambda$.

We will show below that a weight on \mathcal{B}_Λ induces in turn a measure on the infinite path space $\partial\mathcal{B}_\Lambda \cong \Lambda^\infty$; see Theorem 4.9 below for details.

Definition 4.3. An *ultrametric* d on a topological space X is a metric satisfying the strong triangle inequality:

$$d(x, y) \leq \max\{d(x, z), d(y, z)\} \quad \text{for all } x, y, z \in X.$$

Proposition 4.4 ([15, Proposition 2.10]). *Let \mathcal{B}_Λ be a weighted Bratteli diagram with weight w . We define a function d_w on $\partial\mathcal{B}_\Lambda \times \partial\mathcal{B}_\Lambda$ by*

$$d_w(x, y) = \begin{cases} w(x \wedge y) & \text{if } x \neq y \\ 0 & \text{otherwise} \end{cases},$$

where $x \wedge y$ is the longest common initial segment of x and y . (If $r(x) \neq r(y)$ then we say $x \wedge y$ is the empty path \circ , and $w(\circ) = 1$.) Then d_w is an ultrametric on $\partial\mathcal{B}_\Lambda$.

Note that the ultrametric d_w induces the same topology on $\partial\mathcal{B}_\Lambda$ as the cylinder sets in (1); thus, $(\partial\mathcal{B}_\Lambda, d_w)$ is called an *ultrametric Cantor set*.

Definition 4.5. Let A_D be a $D \times D$ matrix with 1 in every entry and let \mathcal{B}_D be the associated Bratteli diagram. Fix $\lambda > 1$, and set

$$d = \ln D / \ln \lambda.$$

We define a weight w_D^λ on the Bratteli diagram \mathcal{B}_D by setting

- (a) $w_D^\lambda(\circ) = 1$.
- (b) For any level 0 vertex $v \in V_0$ of \mathcal{B}_D , $w_D^\lambda(v) = \frac{1}{D}$.
- (c) For any finite path $\gamma \in F^n\mathcal{B}_D$ of length n ,

$$w_D(\gamma) = \lambda^{-n} \frac{1}{D}.$$

According to [15], after choosing a weight on \mathcal{B}_D , we can build a spectral triple associated to it as in the following Theorem. Note that this result is a special case of Section 3 of [15].

Theorem 4.6. Fix an integer $D > 1$ and $\lambda > 1$. Let $(\mathcal{B}_D, w_D^\lambda)$ be the weighted Bratteli diagram with the choice of weight w_D^λ as in Definition 4.5. Let $(\partial\mathcal{B}_D, d_w^\lambda)$ be the associated ultrametric Cantor set. Then there is an even spectral triple $(C_{Lip}(\partial\mathcal{B}_D), \mathcal{H}, \pi_\tau, \not{D}, \Gamma)$, where

- $C_{Lip}(\partial\mathcal{B}_D)$ is the pre- C^* -algebra of Lipschitz continuous functions on $(\partial\mathcal{B}_D, d_w^\lambda)$,
- for each choice function $\tau : F\mathcal{B}_D \rightarrow \partial\mathcal{B}_D \times \partial\mathcal{B}_D$,³ a faithful representation π_τ of $C_{Lip}(\partial\mathcal{B}_D)$ is given by bounded operators on the Hilbert space $\mathcal{H} = \ell^2(F\mathcal{B}_D) \otimes \mathbb{C}^2$ as

$$\pi_\tau(f) = \bigoplus_{\gamma \in F(\mathcal{B}_D)_\circ} \begin{pmatrix} f(\tau_+(\gamma)) & 0 \\ 0 & f(\tau_-(\gamma)) \end{pmatrix};$$

- the Dirac operator \not{D} on \mathcal{H} is given by

$$\not{D} = \bigoplus_{\gamma \in F(\mathcal{B}_D)_\circ} \frac{1}{w_D^\lambda(\gamma)} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix};$$

- the grading operator is given by $\Gamma = 1_{\ell^2(F(\mathcal{B}_D)_\circ)} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Definition 4.7 (cf. [15, Theorem 3.8]). The ζ -function associated to the spectral triple of Theorem 4.6 is given by

$$\zeta_D^\lambda(s) := \frac{1}{2} \text{Tr}(|\not{D}|^{-s}) = \sum_{\gamma \in F(\mathcal{B}_D)_\circ} (w_D^\lambda(\gamma))^s. \quad (11)$$

Proposition 4.8 (cf. [15, Theorem 3.8]). The ζ -function in Equation (11) has abscissa of convergence equal to $d = \ln D / \ln \lambda$.

Proof. By a straightforward calculation we get (if we denote by $F^q(\mathcal{B}_D)_\circ$ the set of paths of length q):

$$\sum_{\gamma \in F(\mathcal{B}_D)_\circ} \left(w_D^\lambda(\gamma) \right)^s = D^{-s} \sum_{q \geq -1} \text{Card}(F^q(\mathcal{B}_D)_\circ) \lambda^{-qs} = D^{-s} \sum_{q \geq -1} D^{q+1} \lambda^{-qs},$$

where $\text{Card}(S)$ denotes the cardinality of the set S . It is clear that this sum converges precisely when D/λ^s is smaller than 1, that is whenever

$$s > \frac{\ln D}{\ln \lambda}.$$

□

³A choice function $\tau : F\mathcal{B}_D \rightarrow \partial\mathcal{B}_D \times \partial\mathcal{B}_D$ is a function that satisfies

$$\tau(\gamma) =: (\tau_+(\gamma), \tau_-(\gamma)) \quad \text{where} \quad d_w(\tau_+(\gamma), \tau_-(\gamma)) = w_D^\lambda(\gamma).$$

It is known that the abscissa of convergence coincides with the upper Minkowski dimension of $\partial\mathcal{B}_D \cong \Lambda_D^\infty$ associated to the ultrametric d_w^λ [20, Theorem 2]. In the self-similar cases (when the weight is given as in Definition 4.5), the upper Minkowski dimension turns out to coincide with the Hausdorff dimension [16, Theorem 2.12]. In particular, when the scaling factor λ is just N , the Hausdorff dimensions of $(\Lambda_D^\infty, d_{w_D^N})$ and \mathbb{S}_A coincide, where we equip \mathbb{S}_A with the metric induced by the Euclidean metric on $[0, 1]^2$.

The Dixmier trace $\mu_D^\lambda(f)$ of a function $f \in C_{\text{Lip}}(\partial\mathcal{B}_D)$ is given by the expression below; see Theorem 3.9 of [15] for details.

$$\mu_D^\lambda(f) = \lim_{s \downarrow d} \frac{\text{Tr}(|\mathcal{D}|^{-s} \pi_\tau(f))}{\text{Tr}(|\mathcal{D}|^{-s})} = \lim_{s \downarrow d} \frac{\text{Tr}(|\mathcal{D}|^{-s} \pi_\tau(f))}{2\zeta_D^\lambda(s)}. \quad (12)$$

In particular the limit given in (12) induces a measure μ_D^λ on $\partial\mathcal{B}_D$ characterized as follows. If $f = \chi_{[\gamma]}$ is the characteristic function of a cylinder set $[\gamma]$, and if $F_\gamma\mathcal{B}_D = \{\eta \in F_\gamma\mathcal{B}_D : \eta = \gamma\eta'\}$ denotes the set of all finite paths with initial segment γ , we have

$$\mu_D^\lambda([\gamma]) = \mu_D^\lambda(\chi_{[\gamma]}) = \lim_{s \downarrow d} \frac{\sum_{\eta \in F_\gamma(\mathcal{B}_D)_\circ} (w_D^\lambda(\eta))^s}{\sum_{\eta \in F(\mathcal{B}_D)_\circ} (w_D^\lambda(\eta))^s}. \quad (13)$$

It actually turns out, as we prove in Theorem 4.9 below, that the measure μ_D^λ on $\partial\mathcal{B}_D$ is independent of λ ; so we will also write, with notation as above

$$\mu_D([\gamma]) = \mu_D(\chi_{[\gamma]}) = \mu_D^\lambda([\gamma]) = \mu_D^\lambda(\chi_{[\gamma]})$$

Moreover, by combining Theorem 3.1 with Theorem 4.9 below, we see that μ_D agrees with the Hausdorff measure of \mathbb{S}_A .

Theorem 4.9. *For any choice of scaling factor $\lambda > 1$, the measure μ_D^λ on $\partial\mathcal{B}_D$ induced by the Dixmier trace agrees with the measure M associated to the infinite path representation of \mathcal{O}_D . Namely, for any finite path $\gamma \in F\mathcal{B}_D$, we have*

$$\mu_D([\gamma]) = \frac{1}{D^{|\gamma|+1}} = M([\gamma]). \quad (14)$$

Proof. Note that, although the proof of this Theorem is very long for the more general case of Cuntz–Krieger algebras (cf. [15, Theorem 3.9]), it considerably simplifies for the case of Cuntz algebras covered here. First note that for the choice of the empty path $\gamma = \circ$ (whose cylinder set corresponds to the whole space), we have

$$f(s) = \frac{\sum_{\eta \in F(\mathcal{B}_D)_\circ} (w_D^\lambda(\eta))^s}{\sum_{\eta \in F(\mathcal{B}_D)_\circ} (w_D^\lambda(\eta))^s} = 1 = \mu_D^\lambda(\Lambda_D^\infty) = M(\Lambda_D^\infty). \quad (15)$$

Now we will compute μ_D^λ , for a finite path $\gamma \neq \circ$ of length n in $F^n\mathcal{B}_D$. Define, according to Equation (13),

$$f(s) = \frac{\sum_{\eta \in F_\gamma\mathcal{B}_D} (w_D^\lambda(\eta))^s}{1 + \sum_{\eta \in F\mathcal{B}_D} (w_D^\lambda(\eta))^s}. \quad (16)$$

Note that in the above expression we isolated the term corresponding to the empty path, for which $\left(w_D^\lambda(\circ)\right)^s = 1^s = 1$. Moreover, since γ is not the empty path, then $\eta = \circ$ does not occur in the sum in the numerator. If $\eta \in F_\gamma \mathcal{B}_D$, then $w_D^\lambda(\eta)^s$ only depends on the length of η , say $|\eta| = n + q$ for some $q \in \mathbb{N}_0$, and hence $w_D^\lambda(\eta) = D^{-1}\lambda^{-(n+q)}$. For $q \in \mathbb{N}_0$, let

$$\begin{aligned} F^q \mathcal{B}_D &= \{\eta \in F \mathcal{B}_D : |\eta| = q\}, \\ F_\gamma^q \mathcal{B}_D &= \{\eta \in F_\gamma \mathcal{B}_D : |\eta| = n + q\}. \end{aligned}$$

Then we can write

$$f(s) = \frac{D^{-s} \sum_{q \in \mathbb{N}_0} \text{Card}(F_\gamma^q \mathcal{B}_D) (\lambda^{-(n+q)})^s}{1 + D^{-s} \sum_{q \in \mathbb{N}_0} \text{Card}(F_\gamma^q \mathcal{B}_D) (\lambda^{-q})^s}.$$

Since the vertex matrix A_D of the Bratteli diagram \mathcal{B}_D has 1 in every entry, every edge in \mathcal{B}_D has D possible edges that could follow it. Also note that $\eta \in F^q \mathcal{B}_D$ has its range in V_0 and its source in V_q , and hence we get

$$\text{Card}(F^q \mathcal{B}_D) = D^{q+1}.$$

But any finite path $\eta \in F_\gamma^q \mathcal{B}_D$ can be written as $\eta = \gamma \eta'$. Since γ is fixed, the number of paths $\eta \in F_\gamma^q \mathcal{B}_D$ is the same as the number of possible paths η' . Since $r(\eta') = s(\gamma)$ is also fixed, we get

$$\text{Card}(F_\gamma^q \mathcal{B}_D) = D^q.$$

By multiplying both numerator and denominator of $f(s)$ by D^s , we obtain

$$\begin{aligned} f(s) &= \frac{D^{-s} \sum_{q \in \mathbb{N}_0} D^q (\lambda^{-(n+q)})^s}{1 + D^{-s} \sum_{q \in \mathbb{N}_0} D^{q+1} (\lambda^{-q})^s} = \frac{1}{\lambda^{ns}} \frac{\sum_{q \in \mathbb{N}_0} D^q \lambda^{-qs}}{D^s + \sum_{q \in \mathbb{N}_0} D^{q+1} \lambda^{-qs}} \\ &= \frac{1}{\lambda^{ns}} \frac{\sum_{q \in \mathbb{N}_0} \left(\frac{D}{\lambda^s}\right)^q}{\left(D^s + D \sum_{q \in \mathbb{N}_0} \left(\frac{D}{\lambda^s}\right)^q\right)}. \end{aligned}$$

Since $s > \frac{\ln D}{\ln \lambda}$, we have $\frac{D}{\lambda^s} < 1$, thus $\sum_{q \in \mathbb{N}_0} \left(\frac{D}{\lambda^s}\right)^q$ converges and is equal to $\frac{1}{1 - \frac{D}{\lambda^s}}$. Thus (again multiplying numerator and denominator of $f(s)$ by $1 - \frac{D}{\lambda^s}$),

$$f(s) = \frac{1}{\lambda^{ns}} \frac{\frac{1}{1 - \frac{D}{\lambda^s}}}{\left(D^s + D \frac{1}{1 - \frac{D}{\lambda^s}}\right)} = \frac{1}{\lambda^{ns}} \frac{1}{\left((1 - \frac{D}{\lambda^s})D^s + D\right)}$$

Now take the limit $s \downarrow d$ and recall that $\lambda^d = D$. So we have $(1 - \frac{D}{\lambda^s}) \rightarrow 0$ and hence

$$\lim_{s \downarrow d} f(s) = \frac{1}{\lambda^{nd}} \frac{1}{D} = \frac{1}{D^n} \frac{1}{D} = \frac{1}{D^{n+1}},$$

which is the desired result by Equation (13). □

4.2 The Laplace–Beltrami operator

In Section 4 of [15], the authors use the spectral triple associated to a weighted Bratteli diagram to construct a non-positive definite self-adjoint operator with discrete spectrum (which they fully describe) defined on the infinite path space of the given Bratteli diagram. Moreover, they show in Theorem 4.3 of [15] that the eigenfunctions of Δ_s form an orthogonal decomposition of the L^2 -space of the boundary.

Therefore, by applying the results of Section 4 of [15] to the spectral triples of Section 4.1 above, we obtain, after we choose a weight w_D^λ on \mathcal{B}_D as in Definition 4.5, a non-positive definite self-adjoint operator Δ_s on $L^2(\partial\mathcal{B}_D, \mu_D)$ for any $s \in \mathbb{R}$, where μ_D is the measure on $\partial\mathcal{B}_D$ given in (13). (Recall that μ_D does not depend on λ). Namely, for any $s \in \mathbb{R}$, the Laplace–Beltrami operator Δ_s on $L^2(\partial\mathcal{B}_D, \mu_D)$ is given by

$$\langle f, \Delta_s(g) \rangle = Q_s(f, g) = \frac{1}{2} \int_E \text{Tr}(|\not{D}|^{-s} [\not{D}, \pi_\tau(f)]^* [\not{D}, \pi_\tau(g)] d\mu_D(\tau), \quad (17)$$

where $\text{Dom } Q_s = \text{span}\{\chi_{[\gamma]} : \gamma \in F\mathcal{B}_D\}$ and Q_s is a closable Dirichlet form, and $\mu_D(\tau)$ is the measure induced by the Dixmier trace on the set E of choice functions.

Moreover, the eigenfunctions of Δ_s form an orthogonal decomposition of $L^2(\partial\mathcal{B}_D, \mu_D)$. In the remainder of this section we give the details of this decomposition and formulas for the eigenvalues. In Section 5 below, we describe the relationship between this orthogonal decomposition and the wavelet decomposition of $L^2(\Lambda^\infty, M)$ computed in [7].

Theorem 4.10. [15, Theorem 4.3] *Let (\mathcal{B}_D, w_D^D) be the weighted Bratteli diagram as in Theorem 4.6. (Note that we made here the choice $\lambda = D$ for simplicity.) Let Δ_s be the Laplace–Beltrami operator on $L^2(\partial\mathcal{B}_D, \mu_D)$ given by (17). Then the eigenvalues of Δ_s are 0, associated to the constant function 1, and the eigenvalues $\{\lambda_\eta\}_{\eta \in (F\mathcal{B}_D)_\circ}$ with corresponding eigenspaces $\{E_\eta\}_{\eta \in (F\mathcal{B}_D)_\circ}$ of Δ_s are given by*

$$\lambda_\circ = (G_s(\circ))^{-1} = \frac{2D}{D-1};$$

$$\lambda_\eta = -2 - 2D^{3-s} \frac{1 - D^{(3-s)|\eta|}}{1 - D^{3-s}} - \frac{2D^{3|\eta|+4}}{(D-1)D^{s(|\eta|+1)}}, \quad \eta \in F(\mathcal{B}_D)$$

with

$$E_\circ = \text{span}\left\{D^{-1}(\chi_v - \chi_{v'}) : v \neq v' \in V_0\right\},$$

$$E_\eta = \text{span}\left\{D^{|\eta|+2}(\chi_{[\eta e]} - \chi_{[\eta e']}) : \right.$$

$$\left. \eta \in F(\mathcal{B}_D), e \neq e', |e| = |e'| = 1, r(e) = r(e') = s(\eta)\right\}.$$

Proof. This follows from evaluating the formulas given in Theorem 4.3 of [15], using Theorem 4.9 above to calculate the measures of the cylinder sets, and recalling that the diameter $\text{diam}[\gamma]$ of a cylinder set is given by the weight of γ .

To be precise, since there are D edges with a given range v , the size of the set

$$\{(e, e') \in \Lambda^1 \times \Lambda^1 : r(e) = r(e') = v, e \neq e'\}$$

is $D(D-1)$ for any vertex v . Therefore, for any path $\eta \in \Lambda$, the constant $G_s(\eta)$ from Theorem 4.3 of [15] is given by

$$G_s(\eta) = \frac{D(D-1)D^{-2(|\eta|+2)}}{2w_D(\eta)^{s-2}} = \frac{(D-1)D^{s(|\eta|+1)}}{2D^{4|\eta|+5}}.$$

Observe that, in the notation of [15], a path of “length 0” corresponds to the empty path \circ , that is, whose cylinder set gives entire infinite path space, and a path of “length 1” corresponds to a vertex. In general, the length of a path in [15] corresponds to the number of vertices that this path traverses; hence a path of length n for them is a path of length $n-1$ for us.

In order to compute the eigenvalues λ_η described in Theorem 4.3 of [15], then, we also need to calculate $G_s(\circ) = G_s(\Lambda^\infty)$. Since the infinite path space has diameter 1 by Proposition 2.10 of [15], we obtain

$$G_s(\circ) = G_s(\Lambda^\infty) = \frac{D(D-1)}{2D^2} = \frac{D-1}{2D}.$$

Now, if we denote the empty path \circ by a path of “length -1 ,” we can rewrite the formula (4.3) from [15] for the eigenvalue λ_η associated to a path η as

$$\begin{aligned} \lambda_\eta &= \sum_{k=-1}^{|\eta|-1} \frac{\frac{1}{D^{k+2}} - \frac{1}{D^{k+1}}}{G_s(\eta[0, k])} - \frac{1}{D^{|\eta|+1}G_s(\eta)} \\ &= \frac{1-D}{D^{\frac{D-1}{2D}}} + \sum_{k=0}^{|\eta|-1} \frac{1-D}{D^{k+2}} \frac{2D^{4k+5}}{(D-1)D^{s(k+1)}} - \frac{2D^{3|\eta|+4}}{(D-1)D^{s(|\eta|+1)}} \\ &= -2 - 2D^{3-s} \frac{1-D^{(3-s)|\eta|}}{1-D^{3-s}} - \frac{2D^{3|\eta|+4}}{(D-1)D^{s(|\eta|+1)}}, \end{aligned}$$

using the notation of Definition 2.3, and the fact that

$$\sum_{k=0}^{|\eta|-1} \frac{2D^{3k+3}}{D^{s(k+1)}} = 2D^{3-s} \sum_{k=0}^{|\eta|-1} \frac{D^{3k}}{D^{sk}} = 2D^{3-s} \frac{1-D^{(3-s)|\eta|}}{1-D^{3-s}}.$$

□

5 Wavelets and eigenfunctions for \mathcal{O}_D

In this section, we connect the eigenspaces E_γ of Theorem 4.10 with the orthogonal decomposition of $L^2(\partial\mathcal{B}_D, M)$ associated to the wavelets constructed in [19] Section 3 (see

also Section 4 of [7]). We begin by describing the wavelet decomposition of $L^2(\partial\mathcal{B}_D, M)$, which is a special case of the wavelets of [19] and [7]. To be precise, the wavelets we discuss here are those associated to the $D \times D$ matrix A_D consisting of all 1's, but the wavelets described in [19] are defined for any matrix A with entries from $\{0, 1\}$.

Let Λ_D denote the directed graph with vertex matrix A_D . In what follows, we will assume that we have labeled the D vertices of Λ_D^0 by $\mathbb{Z}_D = \{0, 1, \dots, D-1\}$, and we will write infinite paths in $\Lambda_D^\infty = \partial\mathcal{B}_D$ as strings of vertices $(i_1 i_2 i_3 \dots)$ where $i_j \in \mathbb{Z}_D$ for all j .

Denote by \mathcal{V}_0 the (finite-dimensional) subspace of $L^2(\partial\mathcal{B}_D, M)$ given by

$$\mathcal{V}_0 = \text{span}\{\chi_{\sigma_i(\partial\mathcal{B}_D)} : i \in \mathbb{Z}_D\}.$$

Define an inner product on \mathbb{C}^D by

$$\langle (x_j), (y_j) \rangle_{PF} = \frac{1}{D} \sum_{j=0}^{D-1} \overline{x_j} y_j. \quad (18)$$

We now define a set of D linearly independent vectors $\{c^j : 0 \leq j \leq D-1\} \subset \mathbb{C}^D$, where $c^j = (c_0^j, \dots, c_{D-1}^j)$, by

$$c_\ell^0 = 1 \quad \forall \ell \in \mathbb{Z}_D,$$

and $\{c^j : 1 \leq j \leq D-1\}$ an orthonormal basis for the subspace $\{(1, 1, \dots, 1)\}^\perp$, with \perp taken with respect to the inner product $\langle \cdot, \cdot \rangle_{PF}$.

We now note that we can write each set $R_{[k]} = \sigma_k(\partial\mathcal{B}_D)$ as a disjoint union:

$$R_{[k]} = \bigsqcup_{j=0}^{D-1} R_{[kj]},$$

where

$$R_{[kj]} = \{(i_1 i_2 \dots i_n \dots) \in \partial\mathcal{B}_D : i_1 = k \text{ and } i_2 = j\}.$$

Thus in terms of characteristic functions,

$$\chi_{R_{[k]}} = \sum_{j=0}^{D-1} \chi_{R_{[kj]}} \quad \text{for } k \in \mathbb{Z}_D.$$

Now, define functions $\{f^{j,k}\}_{j,k=0}^{D-1}$ on $\partial\mathcal{B}_D$ by

$$f^{j,k}(x) = \sqrt{D} \sum_{\ell=0}^{D-1} c_\ell^j \chi_{R_{[k\ell]}}(x).$$

Moreover, since $c_\ell^0 = 1$ for all ℓ , we have

$$f^{0,k} = \sqrt{D} \sum_{\ell=0}^{D-1} c_\ell^0 \chi_{R_{[k\ell]}} = \sqrt{D} \sum_{\ell=0}^{D-1} \chi_{R_{[k\ell]}} = \sqrt{D} \chi_{R_{[k]}}.$$

It follows that

$$\text{span}\{f^{0,k}\}_{k=0}^{D-1} = \text{span}\{\chi_{R_{[k]}}\}_{k=0}^{D-1} = \mathcal{V}_0.$$

Now, we can use the functions $f^{j,k}$ to construct a wavelet basis of $L^2(\partial\mathcal{B}_D, M)$. First, a definition: for any word $w = w_1 w_2 \cdots w_n \in (\mathbb{Z}_D)^n$, write $S_w = S_{w_1} S_{w_2} \cdots S_{w_n}$, where $S_{w_i} \in L^2(\partial\mathcal{B}_D, M)$ is the operator defined in Proposition 2.13.

Theorem 5.1 ([19, Theorem 3.2]; [7, Theorem 4.2]). *Fix an integer $D > 1$. Let $\{S_k\}_{k \in \mathbb{Z}_D}$ be the operators on $L^2(\partial\mathcal{B}_D, M)$ described in Proposition 2.13. Let $\{f^{j,k} : j, k \in \mathbb{Z}_D\}$ be the functions on $\partial\mathcal{B}_D$ defined in the above paragraphs. Define*

$$\mathcal{W}_0 = \text{span}\{f^{j,k} : j, k \in \mathbb{Z}_D, j \neq 0\};$$

$$\mathcal{W}_n = \text{span}\{S_w(f^{j,k}) : j, k \in \mathbb{Z}_D, j \neq 0, \text{ and } w \in (\mathbb{Z}_D)^n\}.$$

Then the subspaces \mathcal{V}_0 and $\{\mathcal{W}_n\}_{n=0}^\infty$ are mutually pairwise orthogonal in $L^2(\partial\mathcal{B}_D, M)$ and

$$L^2(\partial\mathcal{B}_D, M) = \text{span}\left(\mathcal{V}_0 \oplus \left[\bigoplus_{n=0}^\infty \mathcal{W}_n\right]\right).$$

To calculate the functions $S_w(f^{j,k})$, we first observe that

$$S_i \chi_{R_{[k]}} = \sqrt{D} \chi_{R_{[ik]}};$$

consequently, if $w = w_1 w_2 \cdots w_n$,

$$S_w(f^{j,k}) = D^{(n+1)/2} \sum_{\ell=0}^{D-1} c_\ell^j \chi_{[w_1 w_2 \cdots w_n k \ell]}. \quad (19)$$

If we instead write the finite path w as γ , and observe that the edges in E_{n+1} with range k are in bijection with the pairs $(k\ell)_{\ell \in \mathbb{Z}_D}$, we see that for any path $\gamma \in F\mathcal{B}_D$ with $|\gamma| = n-1$,

$$S_\gamma(f^{j,k}) = D^{(n+1)/2} \sum_{e \in E_{n+1}} c_e^j \chi_{[\gamma k e]}. \quad (20)$$

A few more calculations lead us to the following

Theorem 5.2. *Let Λ_D be the directed graph whose $D \times D$ adjacency matrix consists of all 1's. For each $\gamma \in \Lambda$, let E_γ be the eigenspace of the Laplace–Beltrami operator Δ_s described in Theorem 4.10. Then for all $n \geq 0$ we can write*

$$\mathcal{W}_n = \bigoplus_{\gamma \in \Lambda^n} E_\gamma.$$

In particular,

$$L^2(\Lambda^\infty, \mu) = \mathcal{V}_{-1} \oplus \mathcal{W}_{-1} \oplus \left[\bigoplus_{n \geq 0} \bigoplus_{\gamma \in \Lambda^n} E_\gamma \right] = \mathcal{V}_0 \oplus \left[\bigoplus_{n \geq 0} \bigoplus_{\gamma \in \Lambda^n} E_\gamma \right].$$

Moreover, for all $i \in \mathbb{Z}_D$ and all $\gamma \in F\mathcal{B}_D$, the isometry S_i given by

$$S_i f((v_1 v_2 \dots)) = \begin{cases} D^{1/2} f((v_2 v_3 \dots)) & \text{if } v_1 = i, \\ 0 & \text{else.} \end{cases}$$

maps E_γ to $E_{i\gamma}$ unitarily.

Proof. Let $\gamma \in F\mathcal{B}_D$ be a path of length n . Recall that the subspaces E_γ are spanned by functions of the form $\chi_{[\gamma e]} - \chi_{[\gamma e']}$, where $e \neq e'$ are edges in E_{n+1} . In other words, if we write a spanning function $\xi_{e,e'} = \chi_{[\gamma e]} - \chi_{[\gamma e']}$ of E_γ as a linear combination of characteristic functions of cylinder sets, we have

$$\xi_{e,e'} = \sum_{f \in E_{n+1}} d_f \chi_{[\gamma f]}$$

where $d_e = 1, d_{e'} = -1, d_f = 0 \forall f \neq e, e'$. In other words, the vector

$$(d_f)_{r(f)=s(\gamma), f \in E_{n+1}}$$

is in the subspace $(1, 1, \dots, 1)^\perp$ of \mathbb{C}^D which is orthogonal to $(1, 1, \dots, 1)$ in the inner product (18). It follows that $E_\gamma \subseteq \mathcal{W}_n$ whenever $|\gamma| = n$.

Now, Theorem 4.3 of [15] tells us that each space E_γ has dimension $D - 1$. Moreover, there are D^{n+1} paths γ of length n , and $E_\gamma \perp E_\eta$ for all γ, η with $|\gamma| = |\eta|$. Therefore,

$$\dim \left(\bigcup_{|\gamma|=n} E_\gamma \right) = D^{n+1}(D - 1).$$

Similarly, $\dim \mathcal{W}_n = \text{Card}(F^{n-1}\mathcal{B}_D) \text{Card}(\{f^{j,k}\}_{j \neq 0}) = D^n \cdot D(D - 1)$. This equality of dimensions thus implies that

$$\mathcal{W}_n = \bigcup_{|\gamma|=n} E_\gamma \quad \forall n \in \mathbb{N}_0.$$

For the last assertion, we simply observe that S_i is an isometry with $S_i S_i^* = \text{id}|_{E_i}$. \square

5.1 Wavelets on \mathbb{S}_A

Let A be an $N \times N$ $\{0, 1\}$ -matrix with precisely D nonzero entries. In this section we will describe wavelets on \mathbb{S}_A associated to the Cuntz algebra \mathcal{O}_D using the measure-preserving isomorphism between (\mathbb{S}_A, H) and $(\partial\mathcal{B}_D, M)$ described in Theorem 3.1.

Since all edges in Λ_D can be preceded (or followed) by any other edge, this infinite path space corresponds simply to $[0, 1]$ by thinking of points in $[0, 1]$ as infinite sequences in $\{0, \dots, D - 1\}^\mathbb{N}$ and using the D -adic expansion.

The natural correspondence between \mathbb{S}_A and points from $[0, 1]$ in their D -adic expansions is given by labeling the nonzero entries in A by the elements of $\{0, 1, \dots, D - 1\}$,

and then identifying a cylinder set $[(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)]$ in \mathbb{S}_A with the cylinder $[d_1 \dots d_n]$, where $d_i \in \{0, \dots, D-1\}$ is the integer corresponding to A_{x_i, y_i} .

Thus, we obtain wavelets on \mathbb{S}_A by using this identification to transfer the wavelets associated to the infinite path representation of \mathcal{O}_D into functions on \mathbb{S}_A . These wavelets will agree with the eigenfunctions E_γ of the Laplace–Beltrami operator associated to the Bratteli diagram for \mathcal{O}_D , by Theorem 5.2 above.

To be more precise, Theorem 5.2 implies that we can interpret the eigenfunctions of Theorem 4.10 as a wavelet decomposition of $L^2(\partial\mathcal{B}_D, M)$, with

$$E_\gamma = \text{span} \left\{ \frac{1}{M[\gamma e]} \chi_{[\gamma e]} - \frac{1}{M[\gamma e']} \chi_{[\gamma e']} \right\}.$$

Here γ is a finite path in the graph Λ_D associated to \mathcal{O}_D ; writing γ as a string of vertices, equivalently, $\gamma = d_0 d_1 d_2 \dots d_n$ for $d_i \in \{0, \dots, D-1\}$. Thus, if $d_i \in \mathbb{Z}_D$ corresponds to the pair $(x_i, y_i) \in S_A$, and $e, e' \in \{0, \dots, D-1\}$ correspond to the pairs $(z, w), (z', w')$ in the symbol set S_A , the wavelet on $L^2(\mathbb{S}_A, H)$ associated to $\frac{1}{M[\gamma e]} \chi_{[\gamma e]} - \frac{1}{M[\gamma e']} \chi_{[\gamma e']}$ is

$$\begin{aligned} & \frac{1}{H([(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n), (z, w)])} \chi_{[(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n), (z, w)]} \\ & - \frac{1}{H([(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n), (z', w')])} \chi_{[(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n), (z', w')]} \\ & = \frac{1}{D^{n+2}} (\chi_{[(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n), (z, w)]} - \chi_{[(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n), (z', w')]}). \end{aligned}$$

This correspondence allows us to transfer the spaces E_γ from $L^2(\partial\mathcal{B}_D, M)$ to $L^2(\mathbb{S}_A, H)$, giving us an orthogonal decomposition of the latter. Moreover, the “scaling and translation” operators S_i of Theorem 5.2 from the infinite path representation of \mathcal{O}_D transfer (via the same correspondence between pairs (x, y) with $A(x, y) \neq 0$ and elements of $\{0, \dots, D-1\}$) to the operators T_i on $L^2(\mathbb{S}_A, H)$ introduced in Theorem 3.1. In other words, these operators T_i allow us to move between the orthogonal subspaces of $L^2(\mathbb{S}_A, H)$, enabling us to view this as a wavelet decomposition.

6 Spectral triples and Laplacians for the Cuntz algebra \mathcal{O}_D : the uneven weight case

6.1 The spectral triple

We are going to work in the general framework of Section 4 with the difference that the weight (which we call w_D^r) is different from the Perron–Frobenius weights w_D^λ we previously defined in Definition 4.5. For this section, we require that our weight is defined on finite paths as in Definition 4.1, rather than on vertices as in Definition 4.5. In particular, the weight w_D^r will not be self-similar in the sense that $w_D^r(\gamma)$ will not depend only on the length and the source of γ , but also on the precise sequence of edges making up γ .

Definition 6.1. Fix a vector $r = (r_1, \dots, r_D)$ of positive numbers satisfying $\sum_i r_i = 1$. (We also note that this condition is not essential, although it makes a nice normalization.) The weight w_D^r on the graph Λ_D with D vertices v_1, \dots, v_D (equivalently, the Bratteli diagram \mathcal{B}_D) associated to the matrix A_D is defined as follows.

1. Whenever γ is the trivial (empty) path \circ , we set $w_D^r(\circ) = 1$.
2. Associate to each vertex v_i the weight r_i :

$$w_D^r(v) = r_v, \quad \forall v \in \Lambda_0.$$

3. Given a path $\gamma = (e_1 \dots e_n)$ with $|e_j| = 1$, $s(e_i) = v_{j_i}$, and $r(e_1) = v_{j_0}$, we set the weight of γ to be

$$w_D^r(\gamma) = \prod_{i=0}^n r_{j_i}.$$

4. The diameter $\text{diam}[\eta]$ of a cylinder set $[\eta]$ is defined to be equal to its weight,

$$\text{diam}[\eta] = w_D^r(\eta).$$

Note in particular that $[\circ] = \Lambda_D^\infty$ and so $\text{diam}[\circ] = 1$, which is consistent with the choice of our normalization.

The set of finite paths on a graph has a natural tree structure. In fact, if we denote by $(e_1 \dots e_n)$ a string of composable edges (thus requiring $s(e_{i-1}) = r(e_i)$, $\forall i$) then the “parent” of $(e_1 \dots e_n)$ is $(e_1 \dots e_{n-1})$; the root is the path \circ of length -1 which corresponds to Λ_D^∞ . In addition, the weight $w_D^r(\gamma)$ decreases to 0 as the length of γ , $|\gamma|$, increases to infinity. Therefore, the Pearson–Bellissard construction from [20] applies, and there is a spectral triple associated to the set of infinite paths as in Theorem 4.6 (see also [15, 9]). To be more precise, we have:

Proposition 6.2. *Let \mathcal{B}_D be the Bratteli diagram associated to the matrix A_D . Let (\mathcal{B}_D, w_D^r) be the weighted Bratteli diagram given in Definition 6.1. Let $(\partial\mathcal{B}_D, d_w^r)$ be the associated ultrametric Cantor set. Then there is an even spectral triple $(C_{Lip}(\partial\mathcal{B}_D), \mathcal{H}', \pi'_\tau, \not{D}', \Gamma')$.*

The ζ -function associated to the spectral triple of Theorem 4.6 is given by

$$\zeta_D^r(s) = \frac{1}{2} \text{Tr}(|\not{D}'|^{-s}) = \sum_{\lambda \in F(\mathcal{B}_D)^\circ} (w_D^r(\lambda))^s.$$

We now want to compute the abscissa of convergence s_r of the above ζ -function.

Proposition 6.3. *The abscissa of convergence s_r of the ζ -function $\zeta_D^r(s)$ associated to the spectral triple in Proposition 6.2 is 1.*

Proof. The formula for the ζ -function can be written as follows:

$$\zeta_D^r(s) = \frac{1}{2} \text{Tr}(|\mathcal{D}'|^{-s}) = \sum_{n=-1}^{\infty} \sum_{\lambda \in \Lambda^n} \left(w_D^r(\lambda) \right)^s, \quad (21)$$

with the convention that a path of length -1 is the empty path \circ with associated cylinder set Λ^∞ . In order to enumerate how many paths of which weight there are in $F(\mathcal{B})_\circ$, we will use the following argument. Consider the following formal polynomial in D variables X_1, \dots, X_D with integer coefficients:

$$P(X_1, \dots, X_D) = \left(\sum_{i=1}^D X_i \right)^{n+1}.$$

After expanding, each monomial is of the form $c \prod_i X_i^{\alpha_i}$ where c is a constant. The constant c counts how many partitions of $\{0, \dots, n\}$ into D (possibly empty) subsets there are, of cardinality respectively $\alpha_1, \dots, \alpha_D$. The set of such partitions for all possible choices of $\alpha_1, \dots, \alpha_D$ is in bijection with $F^n \mathcal{B}_D$: given $\gamma = (e_1 \dots e_n)$ (with $|e_i| = 1$ and $s(e_{i-1}) = r(e_i)$, $\forall i$), let $U_i = \{j \in \{0, \dots, n\} : s(e_j) = v_i\}$. One sees that $\{U_i\}_{i=1}^D$ defines a partition of $\{0, \dots, n\}$, and the map from $F^n \mathcal{B}_D$, the set of finite paths of \mathcal{B}_D of length n , to the set of such partitions is a bijection. Indeed,

$$w_D^r(\gamma) = \prod_i r_i^{\alpha_i}.$$

Now, we see that the sum in Equation (21) can be rewritten as

$$\zeta_D^r(s) = \sum_{n=-1}^{\infty} P(r_1, \dots, r_D)^{s(n+1)} = \sum_{n=-1}^{\infty} \left(\sum_{i=1}^D r_i^s \right)^{n+1}.$$

This is a geometric series, which converges if and only if $\sum_i r_i^s < 1$. The function $s \mapsto \sum_i r_i^s$ is a decreasing function on \mathbb{R}_+ (since all the r_i are less than 1), and $\sum_i r_i = 1$. Therefore, the abscissa of convergence is exactly $s_r = 1$. \square

Remark 6.4. Note that this guarantees that the upper Minkowski dimension of $(\partial \mathcal{B}_D, d_{w^r})$ is 1, see [20, Theorem 2].

Theorem 6.5. *The measure μ_D^r on $\partial \mathcal{B}_D$ induced by the Dixmier trace is defined by $\mu_D^r([\gamma]) = w_D^r(\gamma)$.*

Proof. Note first that for the case $\gamma = \circ$, the result follows immediately from the definitions of μ_D^r and w_D^r . Given a cylinder set $[\gamma] \neq \Lambda^\infty$, we have

$$\mu_D^r([\gamma]) = \lim_{s \rightarrow s_r^+} \frac{\sum_{\eta: r(\eta)=s(\gamma)} \left(w_D^r(\gamma\eta) \right)^s}{\zeta_D^r(s)}.$$

One remark is in order: if γ is a path of length n and $0 < m < n$, then

$$\begin{aligned}
w_D^r(\gamma) &= w_D^r(r(e_1)) \prod_{i=1}^n w_D^r(s(e_i)) \\
&= \left(w_D^r(r(e_1)) \prod_{i=1}^m w_D^r(s(e_i)) \right) \left(w_D^r(s(e_{m+1})) \prod_{i=m+2}^n w_D^r(s(e_i)) \right) \\
&= \left(w_D^r(r(e_1)) \prod_{i=1}^m w_D^r(s(e_i)) \right) \left(w_D^r(r(e_{m+2})) \prod_{i=m+2}^n w_D^r(s(e_i)) \right) \\
&= w_D^r(e_1 e_2 \cdots e_m) w_D^r(e_{m+2} \cdots e_n).
\end{aligned}$$

In particular, $w_D^r(\gamma\eta)$ is **not** $w_D^r(\gamma)w_D^r(\eta)$. Indeed, any path of the form $\gamma\eta$ with $s(\gamma) = r(\eta)$ can be written uniquely as $\gamma e \eta'$ where e is the unique edge with $r(e) = s(\gamma)$ and $s(e) = r(\eta')$. By the computation above, $w_D^r(\gamma e \eta') = w_D^r(\gamma)w_D^r(\eta')$. Moreover, since Λ_D has precisely one edge connecting any pair of vertices, every finite path η' in Λ gives rise to exactly one e such that $s(e) = r(\eta')$ and $r(e) = s(\gamma)$. Therefore,

$$\sum_{\eta: r(\eta)=s(\gamma)} (w_D^r(\gamma\eta))^s = \sum_{\eta' \in \Lambda} (w_D^r(\gamma))^s (w_D^r(\eta'))^s = (w_D^r(\gamma))^s \alpha(s),$$

where $\alpha(s) = \sum_{\eta' \in \Lambda} (w_D^r(\eta'))^s$. Moreover, since $\lim_{s \rightarrow 1^+} \alpha(s) = +\infty$, we have

$$\mu_D^r([\gamma]) = \lim_{s \rightarrow 1^+} (w_D^r(\gamma))^s \frac{\alpha(s)}{1 + \alpha(s)} = w_D^r(\gamma).$$

□

In particular, we do *not* have $\mu_D^r = \mu_D = M$. This should not be completely surprising, however. The Perron–Frobenius measure $M = \mu_D$ is the unique measure on Λ^∞ under the following assumptions: the measure is a probability measure, and $\mu_D[\gamma]$ only depends on $|\gamma|$ and $s(\gamma)$. The second assumption is not satisfied for μ_D^r .

Note also that the choice of weight w_D^r does not define a self-similar ultrametric Cantor set in the sense of [16, Definition 2.6], since again, the diameter of $[\gamma]$ does not just depend on $|\gamma|$ and $s(\gamma)$ but also on the specific sequence of edges.

6.2 The Laplace–Beltrami Operator

As in Section 4.2, the Dixmier trace associated to the spectral triple of Proposition 6.2 induces the probability measure $\mu_D^r(\tau)$ on the set of choice functions; thus, by the classical theory of Dirichlet forms we can define a Laplace–Beltrami operator Δ_s^r on $L^2(\partial\mathcal{B}_D, \mu_D^r)$ as in Proposition 4.1 of [15] by

$$\langle f, \Delta_s^r(g) \rangle = Q_s(f, g) = \frac{1}{2} \int_E \text{Tr}(|\mathcal{D}|^{-s} [\mathcal{D}, \pi_\tau(f)]^* [\mathcal{D}, \pi_\tau(g)]) d\mu_D^r(\tau), \quad (22)$$

where $\text{Dom } Q_s = \text{span}\{\chi_\gamma : \gamma \in F\mathcal{B}_2\}$ is a closable Dirichlet form.

As before, Δ_s^r is self-adjoint and has pure point spectrum, and we can describe the spectrum of Δ_s^r explicitly. For our case we can additionally compute the eigenvalues and the eigenfunctions of Δ_s^r as follows.

Theorem 6.6. [15, Theorem 4.3] *Let Δ_s^r be the Laplace–Beltrami operator on $L^2(\partial\mathcal{B}_2, \mu_D^r)$ given by (22). Then the eigenvalues $\{\lambda_\eta^r\}$ and corresponding eigenspaces $\{E_\eta^r\}$ of Δ_s^r are given by, for $\eta \in F\partial\mathcal{B}_D$,*

$$\lambda_\eta^r = \sum_{k=-1}^{|\eta|-1} \frac{1}{G_s(\eta[0, k])} \left(\mu_D^r[\eta[0, k+1]] - \mu_D^r[\eta[0, k]] \right) - \frac{\mu_D^r[\eta]}{G_s(\eta)},$$

$$E_\eta^r = \text{span} \left\{ \frac{\chi_{[\eta e]}}{\mu_D^r[\eta e]} - \frac{\chi_{[\eta e']}}{\mu_D^r[\eta e']} : e \neq e', |e| = |e'| = 1, r(e) = r(e') \right\},$$

where $\eta[0, -1] = \circ$ and $\chi_{[\circ]} = \partial\mathcal{B}_D$, $G_s(\eta[0, -1]) = \frac{1}{2} \sum_{v \neq w \in \Lambda^0} \mu_D^r[v] \mu_D^r[w]$, and for $\xi \in F\mathcal{B}_D$,

$$G_s(\xi) = \frac{1}{2} w_D^r(\xi)^{2-s} \sum_{e \neq e' \in r^{-1}(s(\xi))} \mu_D^r[\xi e] \mu_D^r[\xi e'].$$

In addition, 0 is an eigenvalue for the constant function 1, and $\lambda_\circ = (G_s(\circ))^{-1}$ is an eigenvalue with eigenspace

$$E_\circ^r = \text{span} \left\{ \frac{\chi_{[v]}}{\mu_D^r[v]} - \frac{\chi_{[v']}}{\mu_D^r[v']} : v \neq v', v, v' \in V_0 \right\}.$$

Proof. Although Theorem 4.3 of [15] is stated only for the case when the weight function $w(\gamma)$ only depends on the length and the source of the path γ , as in Definition 4.5, a careful examination of the proof of that Theorem will show that the same proof works verbatim in the case of the weight w_D^r . \square

6.3 Eigenvalues and eigenfunctions for the \mathcal{O}_2 case

We are going to explicitly compute here the eigenvalues for the Laplace–Beltrami operator Δ_s^r in the $D = 2$ case. Theorem 6.6 specializes in the case of \mathcal{O}_2 to give

The formulas in Proposition 6.8 below allow us to compute in principle the eigenvalue associated to any finite path. However it seems difficult to get an explicit formula that covers all the cases as the calculations in full generality are difficult to manage because of challenging bookkeeping.

Lemma 6.7. *With notation as above, for a finite path $\xi(p, q) \in F(\partial\mathcal{B}_2)$ having p vertices equal to v_1 and q vertices equal to v_2 we have we have*

$$G_s(\xi(p, q)) = r^{4p+1-ps} (1-r)^{4q+1-qs}$$

More generally, if ξ is any path, one can write

$$G_s(\xi) = (\mu_2^r[\xi])^{4-s} r(1-r).$$

Proof. We start with the second point. If ξ is coded by its vertices, $\xi = (v_0, \dots, v_{|\xi|})$ and $e \neq e'$ are vertices such that $r(e) = r(e') = s(\xi)$, then $w_D^r(\xi e)w_D^r(\xi e') = (w_D^r(\xi))^2 r(1-r)$. Since $\mu_D^r[\xi] = w_D^r(\xi)$, we have

$$G_s(\xi) = \frac{1}{2}(\mu_D^r[\xi])^{2-s} 2(\mu_D^r[\xi])^2 r(1-r)$$

and the result follows. (Note that the factor 2 appears because (v_1, v_2) and (v_2, v_1) are the two pairs in the index of the sum defining $G_s(\xi)$.) For the first point, we compute

$$G_s(\xi(p, q)) = (1/2)[r^p(1-r)^q]^{2-s} 2((r^p(1-r)^q r)(r^p(1-r)^q(1-r))).$$

□

Proposition 6.8. *Let Δ_s^r be the Laplace–Beltrami operator on $L^2(\partial\mathcal{B}_2, \mu_2^r)$ given by (22) for the choice of weight induced by*

$$w_2^r(v_1) = r, \quad w_2^r(v_2) = (1-r),$$

where $r \in [0, 1]$ is fixed. (Note that the notation used above is slightly different from the notation we used in Theorem 6.6). Let $\eta \in F\partial\mathcal{B}_2$ of length n be determined by the string of vertices (v_0, \dots, v_n) ; also we write (v_0, \dots, v_k) for $\eta[0, k]$, for any $k \leq n$. Then we have

$$\lambda_\eta^r = \frac{w_2^r(v_0) - 1}{r(1-r)} + \sum_{k=0}^{n-1} \frac{(\mu_2^r[v_0, \dots, v_k])^{s-3}}{r(1-r)} (w_2^r(v_{k+1}) - 1) - \frac{(\mu_2^r[\eta])^{s-3}}{r(1-r)}.$$

Proof. We will use the fact that if one codes η by its vertices $\eta = (v_0, \dots, v_{|\eta|})$, then $\mu_2^r[v_0, \dots, v_k] = \mu_2^r[v_0, \dots, v_i] \mu_2^r[v_{i+1}, \dots, v_k]$, as was established in the proof of Theorem 6.5. Consequently, we can factor the term $(\mu_2^r[\eta[0, k+1]] - \mu_2^r[\eta[0, k]])$ as follows:

$$\begin{aligned} \mu_2^r[\eta[0, k+1]] - \mu_2^r[\eta[0, k]] &= \mu_2^r[v_0, \dots, v_{k+1}] - \mu_2^r[v_0, \dots, v_k] \\ &= \mu_2^r[v_0, \dots, v_{k+1}](\mu_2^r[v_k] - 1). \end{aligned}$$

We therefore compute

$$\lambda_\eta^r = \sum_{k=-1}^{|\eta|-1} \frac{1}{G_s(\eta[0, k])} (\mu_2^r[\eta[0, k+1]] - \mu_2^r[\eta[0, k]]) - \frac{\mu_2^r[\eta]}{G_s(\eta)},$$

that is (using point 2 of Lemma 6.7)

$$\begin{aligned} \lambda_\eta^r &= \frac{\mu_2^r[v_0] - 1}{G_s(\circ)} \\ &\quad + \sum_{k=0}^{n-1} \frac{1}{r(1-r)(\mu_2^r[v_0, \dots, v_k])^{4-s}} \mu_2^r[v_0, \dots, v_k] (w_2^r(v_{k+1}) - 1) \\ &\quad - \frac{\mu_2^r[\eta]}{r(1-r)(\mu_2^r[\eta])^{4-s}}. \end{aligned}$$

The result follows from algebraic simplifications. Note in particular that $G_s(\circ) = r(1-r)$. □

One can also construct representations and wavelet spaces of \mathcal{O}_2 associated to the weighted Bratteli diagram $(\partial\mathcal{B}_2, w_2^r)$; see Theorem 3.8 of [8]. This is the analogue of Theorem 5.1 above for the uneven weight case. We now compute the eigenspaces corresponding to the eigenvalues of Proposition 6.8 above, and show that they coincide with the wavelet spaces described in [8] Theorem 3.8. In other words, we will show that if $\tilde{\mathcal{W}}_k$ are the orthogonal subspaces of $L^2(\partial\mathcal{B}_2, w_2^r)$ described in Theorem 3.8 of [8],

$$\tilde{\mathcal{W}}_k = \bigoplus_{\eta: |\eta|=k} E_\eta^r.$$

What is done below is similar to the result of Theorem 5.2, but we allow unequal weights in what follows.

By evaluating the formulas given in Theorem 6.6 we obtain:

Proposition 6.9. *Let Δ_s^r be the Laplace–Beltrami operator on $L^2(\partial\mathcal{B}_2, \mu_2^r)$ given by (22) for the choice of weight induced by the choice on the vertices v_1 and v_2 of the associated graph as*

$$w_2^r(v_1) = r_1, \quad w_2^r(v_2) = r_2 = 1 - r_1,$$

where $r = r_1 \in [0, 1]$ is fixed. If we let \circ denote the empty path, then the eigenspace E_\circ^r with eigenvalue λ_\circ^r is given by Λ_2^∞ , hence has dimension 1 and

$$E_\circ^r = \text{span} \left\{ \frac{\chi_{[v_1]}}{\mu_2^r([v_1])} - \frac{\chi_{[v_2]}}{\mu_2^r([v_2])} \right\} = \text{span} \left\{ \frac{\chi_{[v_1]}}{r} - \frac{\chi_{[v_2]}}{1-r} \right\}.$$

Given a finite non-empty path $\eta = v_{j_0}v_{j_1}\dots v_{j_n} \in F\partial\mathcal{B}_2$ with $n+1$ vertices, where $j_i \in \{1, 2\} \forall i$, the eigenspace E_η^r with corresponding eigenvalue λ_η^r described in Proposition 6.8 is given by

$$\begin{aligned} E_\eta^r &= \text{span} \left\{ \frac{\chi_{[\eta e]}}{\mu_2^r[\eta e]} - \frac{\chi_{[\eta e']}}{\mu_2^r[\eta e']} : e \neq e', |e| = |e'| = 1, r(e) = r(e') = s(\eta) \right\} \\ &= \text{span} \left\{ \frac{1}{(\prod_{i=0}^n r_{j_i})r} \chi_{[v_{j_0}v_{j_1}\dots v_{j_n}v_1]} - \frac{1}{(\prod_{i=0}^n r_{j_i})(1-r)} \chi_{[v_{j_0}v_{j_1}\dots v_{j_n}v_2]} \right\}. \end{aligned}$$

We now show how the scaling functions generating \mathcal{V}_0 in Theorem 3.8 of [8] fit into the eigenspace picture described above.

Lemma 6.10. *Let $r \in [0, 1]$ be given, and let μ_2^r be the Markov probability measure on the infinite path space Λ_2^∞ corresponding to the weight assigning r to the vertex v_1 and $1-r$ to the vertex v_2 . Let \mathcal{V}_{-1} denote the space of constant functions on Λ_2^∞ . Then the scaling space $\tilde{\mathcal{V}}_0$ described in Theorem 3.8 of [8] as the span of $\{\chi_{[v_1]}, \chi_{[v_2]}\}$, the characteristic functions of cylinder sets corresponding to the vertices, can be written as*

$$\tilde{\mathcal{V}}_0 = \mathcal{V}_{-1} \oplus E_\circ^r,$$

where E_\circ^r is the eigenspace corresponding to the empty path.

Proof. We note that $\tilde{\mathcal{V}}_0$, being generated by the orthogonal functions $\chi_{[v_1]}$ and $\chi_{[v_2]}$, has dimension 2. On the other hand, the space \mathcal{V}_{-1} of constant functions on Λ_2^∞ has dimension 1 and

$$E_\circ^r = \text{span} \left\{ \frac{\chi_{[v_1]}}{\mu_2^r([v_1])} - \frac{\chi_{[v_2]}}{\mu_2^r([v_2])} \right\}$$

also has dimension 1 and is orthogonal to \mathcal{V}_{-1} . It follows by a dimension count that

$$\mathcal{V}_0 = \mathcal{V}_{-1} \oplus E_\circ^r,$$

as desired. \square

Proposition 6.11. *Let μ_2^r be the Markov probability measure on the infinite path space Λ_2^∞ corresponding to the weight assigning r to the vertex v_1 and $1 - r$ to the vertex v_2 . Then for the corresponding representation of \mathcal{O}_2 on $L^2(\Lambda_2^\infty, \mu_2^r)$ defined in Theorem 3.8 of [8], we have*

$$\tilde{\mathcal{W}}_0 = \text{span}_{\eta: |\eta|=0} \{E_\eta^r\},$$

where E_η^r are the eigenspaces of the Laplace-Beltrami operator defined in Proposition 6.9.

Proof. As in Theorem 3.8 of [8] and Section 5 above, we have an inner product on \mathbb{C}^2 defined by

$$\langle (x_j), (y_j) \rangle = \sum_{j=1}^2 \bar{x}_j \cdot y_j \cdot r_j,$$

and a fixed vector $c^0 = c^{0,k} = (1, 1)$. For $k = 1, 2$, we find an orthonormal basis for $\{c^{0,k}\}^\perp$ denoted by $\{c^{1,k}\}$, where $c^{1,k} = (c_\ell^{1,k})_{\ell \in \{1,2\}}$.

But here, a straightfoward calculation shows that we can take

$$c^{1,k} = \sqrt{r(1-r)} \left(\frac{1}{r}, -\frac{1}{1-r} \right), \quad k = 1, 2.$$

Therefore the wavelet $\psi_{1,k}$ of Theorem 3.8 of [8] is given by

$$\begin{aligned} \psi_{1,k} &= \frac{\sqrt{r(1-r)}}{\sqrt{r_k}} \left[\frac{\chi_{[v_k v_1]}}{r} - \frac{\chi_{[v_k v_2]}}{1-r} \right] \\ &= \sqrt{r(1-r)r_k} \left[\frac{\chi_{[v_k v_1]}}{\mu_2^r([v_k v_1])} - \frac{\chi_{[v_k v_2]}}{\mu_2^r([v_k v_2])} \right]. \end{aligned}$$

Recall that

$$E_{v_k}^r = \text{span} \left\{ \frac{1}{\mu_2^r([v_k v_1])} \chi_{[v_k v_1]} - \frac{1}{\mu_2^r([v_k v_2])} \chi_{[v_k v_2]} \right\}$$

is a one-dimensional subspace of $L^2(\partial \mathcal{B}_2, \mu_2^r)$. Moreover, each vector $\psi_{1,k}$ is evidently a scalar multiple of the single spanning vector from E_η^r for $\eta = v_k$ a path of length 0. Taking the span of the two vectors from $E_{v_1}^r$ and $E_{v_2}^r$ gives exactly the span of the $\psi_{1,k}$ for $k = 1, 2$; since $\tilde{\mathcal{W}}_0$ is defined to be the span of the vectors $\psi_{1,k}$, the result follows. \square

We now relate higher dimensional wavelet subspaces to the corresponding eigenspaces for the Laplacian:

Lemma 6.12. *Let μ_2^r be the Markov probability measure on the infinite path space Λ_2^∞ corresponding to the weight assigning r to the vertex v_1 and $1 - r$ to the vertex v_2 . Then for the corresponding representation of \mathcal{O}_2 on $L^2(\Lambda_2^\infty, \mu_2^r)$ defined in Theorem 3.8 of [8], we have*

$$\tilde{\mathcal{W}}_k = \text{span}_{\eta: |\eta|=k} \{E_\eta^r\},$$

where E_η^r are the eigenspaces of the Laplacian defined in Proposition 6.9.

Proof. We prove the result by induction. We have proved the result for $k = 0$ directly. We now suppose that for $k = n$ we have shown

$$\tilde{\mathcal{W}}_n = \text{span}_{\eta: |\eta|=n} \{E_\eta^r\},$$

where, as defined in Theorem 3.8 of [8],

$$\tilde{\mathcal{W}}_n = \text{span} \{S_w(\psi_{1,k}) : k = 1, 2, w \text{ is a word of length } n\},$$

for $\psi_{1,1}$ and $\psi_{1,2}$ the wavelets of Lemma 6.11, and for $w = w_1 w_2 \cdots w_n$ a word of length n , where $w_i \in \mathbb{Z}_2$, $S_w = S_{w_1} S_{w_2} \cdots S_{w_n}$, where (writing an infinite path x as a sequence of vertices)

$$S_0 f(x) = \begin{cases} r^{-1/2} f(u_2 u_3 \dots) & \text{if } x = (v_1 u_2 u_3 \dots), \\ 0 & \text{else.} \end{cases}$$

and

$$S_1 f(x) = \begin{cases} (1 - r)^{-1/2} f(u_2 u_3 \dots) & \text{if } x = (v_2 u_2 u_3 \dots), \\ 0 & \text{else.} \end{cases}$$

From this and the induction hypothesis, it follows that

$$\begin{aligned} \tilde{\mathcal{W}}_{n+1} &= \text{span} \{S_0(\tilde{\mathcal{W}}_n), S_1(\tilde{\mathcal{W}}_n)\} \\ &= \text{span}_{\eta: |\eta|=n} \{S_0(E_\eta^r), S_1(E_\eta^r)\}, \end{aligned}$$

where a typical element of E_η^r looks like

$$\frac{\chi_{[\eta e]}}{\mu_2^r([\eta e])} - \frac{\chi_{\eta e'}}{\mu_2^r([\eta e'])}.$$

Now if $\eta = u_0 u_1 \cdots u_n$ is a path of length n whose $n + 1$ vertices are given in order by $u_0 u_1 u_2 \cdots u_n$, we compute directly that

$$S_0 \chi_{[\eta]} = \frac{1}{\sqrt{r}} \chi_{[v_1 \eta]},$$

and

$$S_1 \chi_{[\eta']} = \frac{1}{\sqrt{1-r}} \chi_{[v_2 \eta']}.$$

Therefore we can write, for η of length n and e and e' of length 1 with $e \neq e'$,

$$\begin{aligned} S_0\left(\frac{\chi_{[\eta e]}}{\mu_2^r([\eta e])} - \frac{\chi_{[\eta e']}}{\mu_2^r([\eta e'])}\right) &= \frac{1}{\sqrt{r}} \left[\frac{\chi_{[v_1 \eta e]}}{\mu_2^r([\eta e])} - \frac{\chi_{[v_1 \eta e']}}{\mu_2^r([\eta e'])} \right] \\ &= \sqrt{r} \left[\frac{\chi_{[v_1 \eta e]}}{\mu_2^r([v_1 \eta e])} - \frac{\chi_{[v_1 \eta e']}}{\mu_2^r([v_1 \eta e'])} \right] \end{aligned}$$

which is a constant multiple of

$$\frac{\chi_{[v_1 \eta e]}}{\mu_2^r([v_1 \eta e])} - \frac{\chi_{[v_1 \eta e']}}{\mu_2^r([v_1 \eta e'])}$$

which is a spanning function for the one-dimensional subspace $E_{v_1 \eta}^r$. Similarly, $S_1(\frac{\chi_{[\eta e]}}{\mu_2^r([\eta e])} - \frac{\chi_{[\eta e']}}{\mu_2^r([\eta e'])})$ is a constant multiple of

$$\frac{\chi_{[v_2 \eta e]}}{\mu_2^r([v_2 \eta e])} - \frac{\chi_{[v_2 \eta e']}}{\mu_2^r([v_2 \eta e'])},$$

which spans $E_{v_2 \eta}^r$. Since all paths of length $n + 1$ are of the form $v_i \eta$ for some path η of length n and some vertex v_i , with $i = 1, 2$, it then follows that

$$\text{span}_{\eta: |\eta|=n} \{S_0(E_\eta^r), S_1(E_\eta^r)\} = \text{span}_{\eta': |\eta'|=n+1} (E_{\eta'}^r).$$

But this shows that

$$\tilde{\mathcal{W}}_{n+1} = \text{span}_{\eta': |\eta'|=n+1} (E_{\eta'}^r),$$

and the induction step of the proof is complete. □

The above results have established the following:

Theorem 6.13. *Let μ_2^r be the Markov probability measure on the infinite path space Λ_2^∞ corresponding to the weight assigning r to the vertex v_1 and $1 - r$ to the vertex v_2 . Then for the corresponding representation of \mathcal{O}_2 on $L^2(\Lambda_2^\infty, \mu_2^r)$ defined in Theorem 3.8 of [8], we have that the k^{th} -order wavelets defined there are all constant multiples of functions of the form*

$$\frac{\chi_{[\eta e]}}{\mu_2^r([\eta e])} - \frac{\chi_{[\eta e']}}{\mu_2^r([\eta e'])}, \quad |\eta| = k, \quad |e| = |e'| = 1, \quad r(e) = r(e') = s(\eta).$$

As in the case of Lemma 6.11, the constant coefficient needed to transform the wavelet function $S_\eta \psi_{1,k}$ into the spanning function of $E_{\eta v_k}$ can be computed to be

$$\sqrt{r(1-r)} [\sqrt{r}]^j [\sqrt{1-r}]^{k+1-j},$$

where j is the number of v_1 's appearing as vertices in the path ηv_k .

References

- [1] S. Bezuglyi and P. E. T. Jorgensen, *Representations of Cuntz–Krieger algebras, dynamics on Bratteli diagrams, and path-space measures*, in “Trends in Harmonic Analysis and Its Applications”, 57–88, Contemp. Math. **650**, Amer. Math. Soc., Providence, RI, 2015.
- [2] A. L. Carey, J. Phillips, and A. Rennie, *Semifinite spectral triples associated with graph C^* -algebras*, Traces in number theory, geometry and quantum fields, **35–36**, Aspects Math., E38, Friedr. Vieweg, Wiesbaden, 2008.
- [3] E. Christensen and C. Ivan, *Spectral triples for AF C^* -algebras and metrics on the Cantor set*, J. Operator Theory **56** (2006), 17–46.
- [4] E. Christensen, C. Ivan, M. L. Lapidus, and E. Schrohe, *Spectral triples and the geometry of fractals*, J. Noncommut. Geom. **6** (2012), 249–274.
- [5] A. Connes, *Compact metric spaces, Fredholm modules, and hyperfiniteness*, Ergodic Theory Dynamical Systems **9** (1989), 207–220.
- [6] D.E. Dutkay and P.E.T. Jorgensen, *Monic representations of the Cuntz algebra and Markov measure*, J. Funct. Anal. **267** (2014), 1011–1034.
- [7] C. Farsi, E. Gillaspy, S. Kang and J. Packer, *Separable representations, KMS states, and wavelets for higher-rank graphs*, J. Math. Anal. Appl. **434** (2016) 241–270.
- [8] C. Farsi, E. Gillaspy, S. Kang and J. Packer, *Wavelets and graph C^* -algebras*, arXiv:1601.0006.
- [9] C. Farsi, E. Gillaspy, S. Kang, A. Julien, and J. Packer, *Spectral triples on k -graphs*, in preparation.
- [10] M. Goffeng and B. Mesland, *Spectral triples and finite summability on Cuntz–Krieger algebras*, Doc. Math. **20** (2015), 89170.
- [11] A. an Huef, M. Laca, I. Raeburn and A. Sims, *KMS states on the C^* -algebra of a higher-rank graph and periodicity in the path space*, J. Funct. Anal. **268** (2015) 1840–1875.
- [12] J. E. Hutchinson, *Fractals and self-similarity*, Indiana Univ. Math. J. **30** (1981), 713–747.
- [13] A. Jonsson, *Wavelets on fractals and Besov spaces*, J. Fourier Anal. Appl. **4(3)** (1998), 329–340.
- [14] A. Julien and I. Putnam, *Spectral triples for subshifts*, J. Funct. Anal. **270** (2016), 1031–1063.

- [15] A. Julien and J. Savinien, *Transverse Laplacians for substitution tilings*, Comm. Math. Phys. **301** (2011), 285–318.
- [16] A. Julien and J. Savinien, *Embeddings of self-similar ultrametric Cantor sets*, Topology Appl. **158** (2011), 2148–2157.
- [17] K. Kawamura, *The Perron-Frobenius operators, invariant measures and representations of the Cuntz–Krieger algebras*, J. Math. Phys. **46** (2005), 083514, 6 pp.
- [18] A. Kumjian, D. Pask, I. Raeburn, and J. Renault, *Graphs, groupoids, and Cuntz–Krieger algebras*, J. Funct. Anal. **144** (1997), 505–541.
- [19] M. Marcolli and A.M. Paolucci, *Cuntz–Krieger algebras and wavelets on fractals*, Complex Anal. Oper. Theory **5** (2011), 41–81.
- [20] J. C. Pearson, J. V. Bellissard, *Noncommutative Riemannian geometry and diffusion on ultrametric Cantor sets*, J. Noncommut. Geom. **3** (2009), 447–481.
- [21] I. Raeburn *Graph Algebras*, CBMS Reg. Conf. Series in Math., Vol, 103, American Mathematical Society, Providence, RI, 2005, vii+113 pp.
- [22] G. Robertson and T. Steger, *Affine buildings, tiling systems and higher rank Cuntz–Krieger algebras*, Journal für die reine und angewandte Mathematik **513** (1999), 115–144.
- [23] R. Strichartz, *Piecewise linear wavelets on Sierpinski gasket type fractals*, J. Fourier Analysis and Applications **3** (1997), 387–416.
- [24] M.F. Whittaker, *Spectral triples for hyperbolic dynamical systems*, J. Noncommut. Geom. **7** (2013), no. 2, 563–582.

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